# Zariski-etale comparison, cohomology of curves

J'ignore • 6 Oct 2025

Descent datum: $\mathcal{U} = \{T_i \to T\}$ covering family, a descent datum for quasicoherent sheaf is a collection $\mathcal{F}_i$ and isomorphisms $\varphi_{ij}$ between the pullback of $\mathcal{F}_i$ to $T_i \times_T T_j$ and the pullback of $\mathcal{F}_j$ to $T_i \times_T T_j$ via the two projections respectively, satisfying the cocycle condition on any fiber product $T_i \times_T T_j \times_T T_k$. The datum is effective if there exists a quasi-coherent sheaf $\mathcal{F}$ that pull back to $\mathcal{F}_i$ (more precisely we need to specify isomorphism between the pullback of $\mathcal{F}$ and $\mathcal{F}_i$ compatible with $\varphi_{ij}$). If $\mathcal{V}$ is an fpqc covering, then all descent data are effective.

To show that equivalent sites give equivalent category of sheaves, it is easier using the language of sieves, which are saturated covering families, for details see here.

Note that $Spec(k[[x]]) \to Spec(k[x])$ is almost etale except not being locally finite presented; such morphisms are called formally etale.

For an $\mathcal{O}_S$-module $\mathcal{F}$, we can form $\mathcal{F}_{et} := \mathcal{O}_{S_{et}} \otimes_{\iota^* \mathcal{O}_S} \iota^* \mathcal{F}$, where $\iota : S_{et} \to S_{Zar}$ is the obvious map of sites and $S_{et}$ is the etale sheaf represented by $\mathbb{G}_a$. The functor $\mathcal{F} \mapsto \mathcal{F}_{et}$ from the cateogory of $\mathcal{O}_S$-modules to that of $\mathcal{O}_{S_{et}}$-modules is exact because $\mathcal{O}_{S_{et}}$ is a flat over $\iota^* \mathcal{O}_S$ (at the level of stalk this is $\mathcal{O}_{S,s} \to \mathcal{O}^{sh}_{S,\bar{s}}$ which is a flat extension of rings).

We have the following Zariski-Etale comparison morphism:

$$H^*(S, \mathcal{F}) \to H^*_{et}(S, \iota^* \mathcal{F}) \to H^*_{et}(S, \mathcal{F}_{et})$$

where the first map is due to $H^*(S, \mathcal{F})$ being a universal $\delta$-functor. It turns out the for $\mathcal{F}$ quasi-coherent, this is an isomorphism. For the proof see Theorem 2.1 of this note. The idea is that we have etale descent (fpqc descent in fact) for quasi-coherent sheaf, so $\mathcal{F}_{et}(h : U \to S)$ is just $\Gamma(h^* \mathcal{F}, U)$, and the Cech complex for $\mathcal{F}_{et}$ is exact in higher degrees, so the Cech-to-derived spectral sequence used to compute Zariski cohomology also computes etale cohomology. Note that for non-quasi-coherent sheaf like constant abelian sheaf or $\mathbb{G}_m$ we don't have affine vanishing so the Cech complex is useless to compute etale cohomology for those sheaves, and we use short exact sequence to relate etale cohomology of different sheaves instead.

Note that the higher pushforward (in the Zariski site) of finite morphism doesn't vanish in general for non-quasicoherent sheaf, unlike in the etale site. This is because strictly Henselian local rings (the local rings for the étale topology) have no higher cohomology, and that a finite covering of a strictly Henselian local ring is again a finite product of strictly Henselian local rings. However, as we saw above, finite coverings of local rings for the Zariski topology do have higher cohomology. Maybe we should see this as a hint that the Zariski topology does not have a good local theory in the same way that the étale topology does. For an example see this post (pictorially $X$ has nontrivial cohomology since it looks like a circle formed by two closed points and a generic point).

A crucial ingredient to compute $H^i_{et}(X, \mathbb{G}_m)$ for a smooth curve $X$ over an algebraically closed field is Tsen's theorem (the brauer group of any transcendental extension of degree 1 over an algebraically closed field is zero), combined with the fundamental exact sequence (the one giving $H^1_{Zar}(X, \mathbb{G}_m) = Cl(X) \cong Pic(X)$). Note that in the case of Zariski cohomology it is much easier to get the vanishing because the higher cohomology of a constant sheaf vanishes automatically, see here for the detail. For singular $X$ we use normalization to reduce to the smooth case.

Reference:

https://virtualmath1.stanford.edu/~conrad/Weil2seminar/Notes/L4.pdf

https://drive.google.com/file/d/1zF7jia1HOzj9GcVdryFAt6FB6lm5dzz5/view?usp=sharing (calculation of cohomology of curves)

https://stacks.math.columbia.edu/tag/03RH (etale cohomology of $\mathbb{G}_m$), https://stacks.math.columbia.edu/tag/03SB (etale cohomology of torsion abelian sheaves) & https://stacks.math.columbia.edu/tag/03RN (etale cohomology of $\mathbb{Z}/n\mathbb{Z}$)