Dessin d'enfant

J'ignore • 6 Oct 2025

Grothendieck's idea to study $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ is via its action on geometric objects. We need two facts:

1. We have the following exact sequence
$$1 \to \pi_1^{et}(X_{\overline{K}}) \to \pi_1^{et}(X) \to Gal(\overline{K}/K) \to 1$$
where X is a variety over a perfect field K.

2. For $K = \mathbb{Q}$, we have $\pi_1^{et}(X \times_{Spec(\mathbb{Q})} Spec(\mathbb{C})) = \widehat{\pi}_1(X_{top})$

From 1, the adjoint action of the arithmetic fundamental group $\pi_1(X)$ preserves the geometric fundamental group $\pi_1(X_{\overline{K}})$, and it descends to a map $Gal(\overline{K}/K) \to Out(\pi_1(X_{\overline{K}}))$. For example, if $X = \mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m$, then $\pi_1^{et}(X) \cong \widehat{\mathbb{Z}}$, and the action $\chi : G_{\mathbb{Q}} \to Aut(\widehat{\mathbb{Z}}) \cong \widehat{\mathbb{Z}}^\times$ is the cyclotomic character.

If we consider $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $K = \mathbb{Q}$, then we have a group homomorphism

$$Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to Out(\widehat{F_2})$$

where F_2 is the free group on two generators. The amazing thing is that this map is injective, first shown by Belyi (see this post for a proof of the original statement; the faithfulness of the action of the absolute Galois group can be shown by considering elliptic curves E_j, see Theorem 2 of this article; For generalization of Belyi's theorem, see here).

Drinfeld gives an explicit description of a subgroup $\widehat{GT}$ of $out(\widehat{F_2})$ that conjectually is the image of the absolute Galois group. See Willwacher's note Definition 1.2. The definition is as follows: Let x, y be the two generators of F_2, $\widehat{GT}$ consists of all φ such that $\varphi(x) = x^\lambda, \varphi(y) = f^{-1}y^\lambda f$ such that $\lambda \in 1 + 2\widehat{\mathbb{Z}}$, $f \in [\widehat{F_2}, \widehat{F_2}]$ satisfying

1. $f(y, x) = f(x, y)^{-1}$.
2. $f(z, x)z^m f(y, z)y^m f(x, y)x^m = 1$ if $m = \frac{1-\lambda}{2}$ and $xyz = 1$.
3. $f(x_{12}, x_{23}x_{24})f(x_{13}x_{23}, x_{34}) = f(x_{23}, x_{34})f(x_{12}x_{13}, x_{24}x_{34})f(x_{12}, x_{23})$ where x_{ij} is generator of PB_4.

Reference:

https://fr.wikipedia.org/wiki/Dessin_d%27enfant_(math%C3%A9matiques) (for explanation why dessin d'enfant corresponds to finite covers of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

https://mathoverflow.net/questions/1909/what-are-dessins-denfants

https://swc-math.github.io/aws/2005/05SchnepsNotes.pdf

https://drive.google.com/file/d/1M31uE5uReuX66pJ_KDsunz9mZAsTPsiE/view?usp=sharing