Etale sites are topological invariants

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Preliminary lemma: Intergral iff affine and universally closed (intuitively, integral means that fibers are zero-dimensional; affineness are important, otherwise the fibers could be positively dimensional, c.f. proper morphisms which are almost never affine). For a formal proof, see here.

Universal homeomorphism iff universal injective (radical), surjective (implies universally surjective since surjectivity is preserved under base change), integral (see here for reference).

One important thing to keep in mind is that even though two sites (Grothendieck topology) are not equivalent, they might have the same sheaf theory, and therefore same theory of abelian sheaf cohomology, e.g. the topological etale site X_{et} and the ordinary topological site X_{top} are not equivalent (the former has nontrivial automorphisms), but the category of sheaves $Et(X)$ and $Top(X)$ are.

If $f : X \to X'$ is a universal homeomorphism, then it is not hard to see that the natural transformation $id \to f_* f^*$ and $f^* f_* \to id$ are isomorphisms. However, it turns out that even the etale sites of X and X' are equivalent. For the proof see here for the special case that S are closed subschems of S' with the same underlying space and here for the general case using descent. The etale assumption ensures that $X \to X \times_S X$ is an open immersion (indeed unramifiedness suffices), and we can use the amazing property of open immersion that $X \to Y$ factorizes through an open subscheme $U \subset Y$ iff the set-theoretic image lands in the underlying set of U.

Reference:

https://virtualmath1.stanford.edu/~conrad/Weil2seminar/Notes/etnotes.pdf, 1.1.6.4

https://virtualmath1.stanford.edu/~conrad/Weil2seminar/Notes/L2.pdf