Slick way of proving Kunneth formula

J'ignore • 26 Sep 2025

The aim is to prove Kunneth formula, namely if $H^*(Y)$ is finitely generated and free R-module, then $H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$. We first show it for X, Y CW complexes. For this there is a slick way to do it, and it hinges on the fact that the category of CW complexes are generated by taking homotopy colimit from points, so it boils down to just checking Kunneth formula holds for $Y = *$, which is trivial.

To be more precise about this, we need the notion of a general cohomology theory. The key lemma is the following:

> If there is a natural transformation from one cohomology theory to another that commutes with boundary maps and agree on points, then it is a natural isomorphism on CW complexes.

The idea is to induct on skeletons and 5-lemma. Note that for any cohomology theory we have $h^m(D^n, S^{n-1}) \xrightarrow{\cong} h^{m-n}(*, \emptyset)$ and $\Pi(D^n, S^{n-1}) \to (X^{[n]}, X^{[n-1]})$ induces an isomorphism between h^m. This does the job for finite-dimensional CW complexes.

For infinite dimensional CW complexes, we can use Milnor sequence: If $X = \cup X_n$, we can understand $h^m(X)$ built out of $h^m(X_n)$. Assuming $X_n \subseteq X_{n+1}$, then we have a map $h^m(X) \to \varprojlim h^m(X_\bullet)$. Define $\varprojlim^1 M_i \xrightarrow{\varphi_i} M_{i+1}$ to be the cokernel of $1 - \varphi : \prod M_i \to \prod M_i$ (note that $\varprojlim M_i$ is the kernel of $1 - \varphi$).

There is a short exact sequence $0 \to \varprojlim^1 h^{m-1}(X_\bullet) \to h^m(X) \to \varprojlim h^m(X_\bullet) \to 0$. Let us sketch the proof: Consider the infinite mapping telescope (infinite wedding cake), T, which is homotopy equivalent to X (This uses the homotopy extension property of CW pairs). The idea is that even we don't know how to do infinite Mayer-Vietoris, we can use mapping telescope to reduce to doing finite Mayer vietoris. Consider the even piece T^e and the odd piece T^o. Each of these piece is disjoint union of infinitely many cylinders, and we can use the infinite disjoint union axiom to calculate the cohomology of T^e and T^o. Their intersection is a disjoint union of

X_n indexed by $n+1$. The map of $T^e \cap T^o$ to T^e is inclusion while to T^o is the identity. Now use the Mayer-Vietoris. Note that for homology the higher $\varprojlim^1$ vanishes, see this book, page 121, Prop. 7.53.

Finally, since any space X is weakly homotopy equivalent to a CW-complex by the CW approximation theory, and practically any cohomology theory descends to the weak homotopy category.

Now we use the key lemma to prove Kunneth formula. if Y is a space, then $(X, A) \mapsto H^*(X \times Y, A \times Y)$ is a cohomology theory, so is $(X, A) \mapsto H^*(X, A) \otimes H^*(Y)$ if $H^k(Y)$ is finitely generated free module for each k (finitely generated is needed for the infinite product axiom).