Gelfand trick

J'ignore • 26 Sep 2025

Suppose $H \subset G$ finite groups and π is an irreducible representation of H. When is $Ind_H^G \pi$ multiplicity-free (c.f. Gelfand pair)? Schur's lemma tells us that this holds iff $End(Ind_H^G \pi)$ is commutative. If π is the trivial representation, then the induced representation is essentially $\mathbb{C}[G/H]$, and the endomorphism algebra is the double coset algebra $\mathbb{C}[H \setminus G/H]$. For general representation π, we define the Hecke algebra

$$\mathcal{H} = \{\varphi : G \to End_{\mathbb{C}}(V) : \varphi(kgh) = \pi(k) \circ \varphi(g) \circ \pi(h), \forall g \in G, k, h \in H\}.$$

This is easily seen isomorphic to the endomorphism algebra of $Ind_H^G \pi$ and reduced to the double coset algebra if π is trivial. Thus we would like to seek a criterion that enables us to show that $\mathcal{H}$ is commutative. A group is commutative iff $x \mapsto x^{-1}$ is a homomorphism. The key feature is that this is an involution. We have the following criterion (Theorem 1 of this article):

> Suppose that there is an involution (i.e. anti-homomorphism) $\iota : G \to G$ such that $HgH = H\iota(g)H$ for all $g \in G$ and $\pi(\iota(h)) = \pi(h)$ (This ensures that ι induces an involution $\tilde{\iota}$ on $\mathcal{H}$). Then (G, H) is a Gelfand pair.

The main goal is to show that the Gelfand-Graev representation $Ind_U^G \psi$ is multiplicity-free. The irreducible representations of this representation are called generic and include most of the cuspidal representations that are hard to access by means of parabolic induction (those are the representations π whose restriction to U contains the trivial representation).

The key tool is the Bruhat decomposition, from which we can easily deduce the following variant:

> We have the following decomposition $GL_n(\mathbb{F}_q) = \sqcup_{m \in M} UmU$ where M is the set of monomial matrices (i.e. consist of exactly one nonzero entry in each row and each column).

This follows from the fact that $B = DU = UD$ where D is the diagonal torus. Now if we take $\psi := \chi(x_{12} + x_{23} + ...)$ where $\chi : \mathbb{F}_q \to S^1$ is a nontrivial character, then there is an obvious involution on G, which is reflecting about the anti-diagonal. The hard part is to check that $U\iota(m)U = UmU$ for a double coset representative in the Hecke algebra. For details see Theorem 3 of the above article.