Homotopy theory intro and Model category

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Note that $(X, A) \to (Y, B) \xrightarrow{i} (Cf, Cf|_A)$ is coexact, since

$$Map_*(Cf, W) = \{h : Y \to W + \text{base point preserving null-homotopy } X \xrightarrow{h \circ f}$$

We can iterate this construction, forming the mapping cone of mapping cone, etc. Note that $Ci \simeq \Sigma X$, so the LES is just forming iterated suspension.

1. Cofibrations are good embeddings and satisfies the homotopy extension property. The prototypical examples are inclusion to mapping cylinder $X \to M_f$ for $f : X \to Y$. Equivalently, (X, A) has HEP iff $X \cup_A A \times I$ is a retract of $X \times I$. Since $Top(A \times I, Y) \cong Top(A, Top(I, Y))$ (since I is compact the Hom-tensor adjunction works fine), it is the same as the following diagram.

2. Fibrations are like fiber bundles and satisfies the homotopy lifting property. The prototypical examples are the path space fibration $E_f : PX \to X$ and pullback $E_f \to Y$ along any continous map $f : Y \to X$.

3. Essentially the theory of model category gives something like factorization system (surjectives follow by injectives) but slightly weaker (not requiring the factorization to be functorial), i.e. the notion of weak factorization system. Via it we can formulate the notion of a model category succinctly.

We can check that (X, CX) has HEP, hence so is (Y, Cf) by closure under pushout. We can use it to show that $CY \to Ci$ is a homotopy equivalence by proving a more general lemma that if A is contractible and (A, X) has HEP, then $X \to X/A$ is a homotopy equivalence.

Where does group structure of higher homotopy group come from? We can check S^1 is a cogroup object in the homotopy category (co-H-group), and that if X is any pointed space and Y is any co-H-group, so is $X \wedge Y$ (the smash product). This is similar to the fact that $Func(S, G)$ is a group for any set S and group G.

Recall the Eckmann-Hilton argument: If X is a co-H-group in two ways, say μ and μ', and μ is a cohomomorphism for the μ'-structure (and vice versa) then $\mu = \mu'$ and μ is cocommutative. A corollary is that $\Sigma^2 Y$ is a cocommutative group. The idea behind the Eckmann-Hilton argument is the very simple observation that group objects in the category of groups are precisely abelian groups. For more on Eckmann-Hilton argument, see this.

We see that the homotopy category is very closed to being a triangulated category, except that Σ is not generally invertible (e.g. see this for an example). Inverting Σ gives the category of spectra which is triangulated.

Since CX has a ΣX-coaction, i.e. there is a map $CX \to CX \vee \Sigma X$ by crushing the middle circle of the cone. As a consequence, $[Cf, Z]$ is a $[\Sigma X, Z]$-set, and the map $[Cf, Z] \to [\Sigma X, Z]$ is a map as $[\Sigma X, Z]$-sets.

Definition of homotopy group: Starting from $(S^0, *) \to (S^0, S^0) \to (D^1, S^0) \to (S^1, *) \to (S^1, S^1) \to \ldots$ Define $\pi_k(X, *) = [(S^k, *), (X, *)]$ and $\pi_k(X, A, *) = [(D^k, S^{k-1}), (X, A)]$, and note that $[(S^k, S^k), (X, A)] = \pi_k(A, *)$.

One can show $\pi_{<n}(S^n) = \{0\}$ by perturbation argument. By the path lifting property, $\pi_k(\tilde{X}) \to \pi_k(X)$ is bijective for $k \geq 1$. In particular, the higher homotopy groups of all orientable surface vanish.

Interpretation of $\pi_n(X, A)$: If $(D^n, S^{n-1}) \to (X, A)$ represents zero in $\pi_n(X, A)$, then $F \simeq_{S^{n-1}} F'$ where $F' : (D^n, S^{n-1}) \to (A, A) \to (X, A)$. (Proof is by interpreting the homotopy $F \simeq c_*$ as a homotopy from inverted can to (X, A).)