Homotopy theory intro and Model category

J'ignore • 24 Sep 2025

Note that $(X, A) \to (Y, B) \xrightarrow{i} (Cf, Cf|_A)$ is coexact, since

$Map_*(Cf, W) = \{h : Y \to W + \text{base point preserving null-homotopy } X \xrightarrow{h \circ f}$

We can iterate this construction, forming the mapping cone of mapping cone, etc. Note that $Ci \simeq \Sigma X$, so the LES is just forming iterated suspension.

1. Cofibrations are good embeddings and satisfies the homotopy extension property. The prototypical examples are inclusion to mapping cylinder $X \to M_f$ for $f : X \to Y$. Equivalently, (X, A) has HEP iff $X \cup_A A \times I$ is a retract of $X \times I$. Since $Top(A \times I, Y) \cong Top(A, Top(I, Y))$ (since I is compact the Hom-tensor adjunction works fine), it is the same as the following diagram.

2. Fibrations are like fiber bundles and satisfies the homotopy lifting property (more precisely it is called Hurewicz fibrations). The prototypical examples are the path space fibration $E_f : PX \to X$ and pullback $E_f \to Y$ along any continous map $f : Y \to X$. A Serre fibration is just having the homotopy lifting proprety with respect to $I^n \to I^n \times I$.

We can show that a Serre fibration has the HLP for all CW complexes by induction on skeletons using the fact that $(I^{k-1} \times 0, I^k)$ is homeomorphic to the pair $(D^{k-1} \times 0 \cup S^k \times I, D^{k-1} \times I)$.

If $A \to X$ is a cofibration, then $B^X \to B^A$ is a Serre fibration (use the tensor-hom adjunction). Corollary: $B^I \to B$ is a fibration.

3. Essentially the theory of model category gives something like factorization system (surjectives follow by injectives) but slightly weaker (not requiring the factorization to be functorial), i.e. the notion of weak factorization system. Via it we can formulate the notion of a model category succinctly.

We can check that (X, CX) has HEP, hence so is (Y, Cf) by closure under pushout. We can use it to show that $CY \to Ci$ is a homotopy equivalence by proving a more general lemma that if A is contractible and (A, X) has HEP, then $X \to X/A$ is a homotopy equivalence.

Where does group structure of higher homotopy group come from? We can check S^1 is a cogroup object in the homotopy category (co-H-group), and that if X is any pointed space and Y is any co-H-group, so is $X \wedge Y$ (the smash product). This is similar to the fact that $Func(S, G)$ is a group for any set S and group G.

Recall the Eckmann-Hilton argument: If X is a co-H-group in two ways, say μ and μ', and μ is a cohomomorphism for the μ'-structure (and vice versa) then $\mu = \mu'$ and μ is cocommutative. A corollary is that $\Sigma^2 Y$ is a cocommutative group. The idea behind the Eckmann-Hilton argument is the very simple observation that group objects in the category of groups are precisely abelian groups. For more on Eckmann-Hilton argument, see this.

We see that the homotopy category is very closed to being a triangulated category, except that Σ is not generally invertible (e.g. see this for an example). Inverting Σ gives the category of spectra which is triangulated.

Since CX has a ΣX-coaction, i.e. there is a map $CX \to CX \vee \Sigma X$ by crushing the middle circle of the cone. As a consequence, $[Cf, Z]$ is a $[\Sigma X, Z]$-set, and the map $[Cf, Z] \to [\Sigma X, Z]$ is a map as $[\Sigma X, Z]$-sets.

Construction of relative homotopy sequence: Starting from $(S^0, *) \to (S^0, S^0) \to (D^1, S^0) \to (S^1, *) \to (S^1, S^1) \to ...$ Define $\pi_k(X, *) = [(S^k, *), (X, *)]$ and $\pi_k(X, A, *) = [(D^k, S^{k-1}), (X, A)]$, and note that $[(S^k, S^k), (X, A)] = \pi_k(A, *)$. Alternatively, we can start with the puppe sequence (using mapping fiber rather than mapping cofiber)

One can show $\pi_{<n}(S^n) = \{0\}$ by perturbation argument. By the path lifting property, $\pi_k(\tilde{X}) \to \pi_k(X)$ is bijective for $k \geq 1$. In particular, the higher homotopy groups of all orientable surface vanish.

Interpretation of $\pi_n(X, A)$: If $(D^n, S^{n-1}) \to (X, A)$ represents zero in $\pi_n(X, A)$, then $F \simeq_{S^{n-1}} F'$ where $F' : (D^n, S^{n-1}) \to (A, A) \to (X, A)$. (Proof is by interpreting the homotopy $F \simeq c_*$ as a homotopy from inverted can to (X, A).) Corollary: If (X, A) is a CW pair and (X, A) has an n-cell, then $\pi_n(Y, B) = 0$ implies any map $f : (X, A) \to (Y, B)$ is homotopic rel A to a map $\tilde{f} : (X, A) \to (B, B)$ (The advantage of working relatively is that we can proceed to build the homotopy inductively, and it passes to limit). Corollary of corollary: If $X \subseteq Y$ is an inclusion of subcomplex (both path-connected), and $\pi_n(Y, X) = 0$ for all n, then $i : X \to Y$ admits a deformation retract (relative to X). Final corollary: (Whitehead theorem) $f : X \to Y$ CW complexes is a homotopy equivalence iff $f_* : \pi_k(X, x) \to \pi_k(Y, f(x))$ is an isomorphism for all k. We use CW approximation to replace f by a homotopic map $\tilde{f}$ that is a

cellular map. This gurantees that we can put a CW structure on $M(f)$ such that $X \to M(f)$ is an inclusion of CW subcomplex, and it reduces to showing $X \to M(f)$ is a homotopy equivalence. The relative homotopy long exact sequence shows that $\pi_k(X, x) \to \pi_k(M(f), x)$ is an isomorphism for all k, hence we are done. In particular, if a CW complex has homotopy groups all zero, then it is contractible. This shows that CW complexes are very special objects.

A remark is that an analogous therem holds for chain complexes, if we have $f : C_\bullet \to D_\bullet$ chain complexes of projectives, then f is a chain homotopy equivalence iff $f_* : H_k(C_\bullet) \to H_k(D_\bullet)$ is a quasi-isomorphism (This has to do with the fact that there is a model category structure whose class of cofibrations are maps that are monomorphisms in each degree with projective cokernel).

It is a fact from point set topology that on a paracompact base, a local Serre fibration is a Serre fibration. As a corollary, fiber bundles provide a rich source of Serre fibrations. If $E \to B$ is a surjective fibration, then for $e \in E$, $A \subset B$ implies the map $\pi_n(E, p^{-1}(A), e) \to \pi_n(B, A, p(e))$ is an isomorphism for all n (in other words, the fibration assumption implies that mapping fiber has the same homotopy type as the actual fiber). This implies the homotopy sequence of a fibration (in particular recover result for covering space).

The fundamental group of the base acts on every algebraic invariant attached to the homotopy type of the fiber. There is another description of the monodromy action. Let $f \in \pi_n(B, b)$ represented by $(D^n, S^{n-1}) \to (B, b)$ and γ is a path from b to b', then we can just imagine growing D^n radially by γ.

Recall $\pi_n(\tilde{X}, e) \cong \pi_n(X, p(e))$ for $n \geq 2$. This is actually an $\pi_1(X, x)$ equivariant isomorphism. Fact: $\pi_2(\bigvee_I S^2) \cong \bigoplus \pi_2 S^2 \cong \bigoplus_I \mathbb{Z}$. This implies that $\pi_2(S_2 \wedge S_1) \cong \mathbb{Z}[\mathbb{Z}]$ (action by π_1 is by the natural shift).

Reference:

motivation for higher topos theory: https://mathoverflow.net/questions/433554/higher-topos-theory-whats-the-moral

How to think about model cateogory: https://mathoverflow.net/questions/287091/why-do-we-need-model-categories