automorphy lifting lecture 7-9

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The space of cuspidal automorphic forms $\mathcal{A}_0(GL_n(F) \setminus GL_n(\mathbb{A}_F))$ is the set of all (complex-valued) functions $\varphi : GL_n(F) \setminus GL_n(\mathbb{A}_F) \to \mathbb{C}$ such that

- φ is smooth (i.e. locally constant in the finite places and smooth in the infinite places);
- φ is K-finite/admissible (The space of right translates under right translates by product of maximal compact subgroups (speficied as follows) is finite-dimensional. For the finite places, we use $GL_n(\widehat{\mathcal{O}_F})$ where $\widehat{\mathcal{O}_F}$ is the profinite completion of $\mathcal{O}_F$, so it is $\prod_v \mathcal{O}_{F,v}$. For the infinite places, we use $U_\infty = \prod_{v|\infty} U_v$ where U_v is a maximal compact subgroup of $GL_n(F_v)$, required to be $O(n)$ when v is real and $U(n)$ when v is complex.);
- φ is $\mathfrak{h}$-finite (should be treated together with the previous condition at infinite place; here $\mathfrak{h}$ is the center of the universal envelopping algebra $\mathcal{U}(\mathfrak{g})$ where $\mathfrak{g} := Lie(GL_n(F_\infty)) \otimes \mathbb{C}$ and the action is by $(X\varphi)(g) = \frac{d}{dt}|_{t=0}(\varphi(g\exp(tX)))$ and extended to $\mathcal{U}(\mathfrak{g})$ by universal property, c.f. the Harish-Chandra isomorphism);
- φ is slowly increasing (polynomial growth);
- φ is cuspidal (integral of the left-translates $\varphi(ug)$ along every unipotent radical of the standard parabolic subgroup vanishes);

The space $\mathcal{A}_0(GL_n(F) \setminus GL_n(\mathbb{A}_F))$ is not quite a representation of $GL_n(\mathbb{A}_F)$, because U_∞-finite is not preserved under right translation by g (instead it is $gU_\infty g^{-1}$-finite; note that there is no problem at finite places). However it does admits an action by $GL_n(\mathbb{A}_F^\infty) \times U_\infty$ and an action by $\mathfrak{g}$, and they are related by

$$g(X\varphi) = (ad(g_\infty)X)(g\varphi).$$

Another remark is that in the non-Archimedean case requring K-finite is the same as admissibility (the space of fixed vectors of any compact open subgroup is finite-dimensional), and the latter is more convenient since we don't need to keep track of isotypic components (see Getz, Intro to Automorphic representations, Prop. 5.3.11).

A third remark is that automorphic representations are factorizable, i.e. an irreducible $\pi = \otimes'_v \pi_v$ (Flath's theorem, see Theorem 5.7.1 for a proof).

The center $\mathfrak{h} = \otimes_{\tau:F\to\mathbb{C}} \mathfrak{h}_\tau$ of the universal envelopping algebra at infinite places act by π by scalars. By the Harish-Chandra isomorphism, we have $\mathfrak{h}_\tau \cong \mathbb{C}[x_1, ...x_n]^{S_n}$, so each τ gives us n complex numbers. We make the following definition: If $HC_\tau \subseteq \mathbb{Z}$ for each τ, then it is algebraic. If it has n distinct elements for each τ, then it is regular. The regular algebraic representations are accessible via topology since they appear in the Betti cohomology of the symmetric space $GL_n(F) \setminus GL_n(\mathbb{A}_F)/K$ for some choice of compact subgroup K.

The case $n = 1$ of Global Langlands is a reformulation of class field theory. A cuspidal automorphic representation of GL_1 is just a continuous character $\chi : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times$. The algebraicity condition says that $\chi|_{(F_\infty^\times)^\circ}$ looks like $x \mapsto \prod_{\tau:F\to\mathbb{C}} \tau(x)^{-n_\tau}$ for some integers n_τ. From χ we would like to produce a Galois representation. First we define $\tilde{\chi} : \mathbb{A}_F^\times \to \mathbb{C}^\times$ by $\tilde{\chi} := \chi(x) \prod_{\tau:F\to\mathbb{C}} \tau(x)^{n_\tau}$ (note this no longer trivial on $F^\times$, but it takes $F^\times$ to $\overline{\mathbb{Q}}^\times$). By continuity of the character, it is invariant by some open compact subgroup $U \subset \mathbb{A}_F^\times$ and also on $(F_\infty^\times)^\circ$ by construction. A fundamental fact is that any quotient $\mathbb{A}_F^\times / F^\times U (F_\infty^\times)^\circ$ is finite, so $\tilde{\chi}$ is valued in $\overline{\mathbb{Q}}^\times$ on the entire $\mathbb{A}_F^\times$.

We can now use the isomorphism $\eta : \mathbb{C} \to \overline{\mathbb{Q}_p}$ (restricted to $\overline{\mathbb{Q}}^\times$) to make $\tilde{\chi}$ valued in $\overline{\mathbb{Q}_p}$ and then modify it at the places above p to make it invariant by $F^\times$ by undoing the integral twist:

$$\chi^{(p)} = \eta \circ \tilde{\chi}(x) \prod_{\tau:F\to\overline{\mathbb{Q}_p}} \tau(x_p)^{-n_{\eta^{-1}\circ\tau}}$$

Since this involves places above p, the character $\chi^{(p)}$ will factor through $\mathbb{A}_F^\times / \overline{F^\times (F_\infty^\times)^\circ} \xrightarrow[Art_F]{\cong} G_F^{ab}$.

Similarly, starting from an algebraic p-adic Hecke character, we can get a complex valued algebraic Hecke character. One thing to note is that the image of $\tilde{\chi}$ lies in a number field. First, the image of $\mathbb{A}_F^\times$ under $\prod_{\tau:F\to\overline{\mathbb{Q}_p}} \tau(x)^{n_\tau}$ lies in F^{Gal}, the Galois closure of F (the image of F^{Gal} is independent of choice of the embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}$). At other places the image of $\tilde{\chi}$ is unchanged. For the infinite place, we must have $\chi|_{(F_\infty)^{\times,\circ}} = 1$ since the target is totally disconnected. For finite places $v \nmid p$, the incompatibility of the profinite topologies implies that there is an open neighborhood of 1 such that χ is trivial. Hence by compactness of $\mathcal{O}_{F_v}^\times$, the image $\chi(\mathcal{O}_{F_v}^\times)$ is finite, hence it has image in roots of unity μ_∞. Since $U_1 \cap \mu_\infty = \{1\}$ where U_1 is the open subgroup $1 + \varpi\mathcal{O}_{\overline{\mathbb{Q}_p}}$. Thus $ker(\chi|_{\prod_{v\nmid p} \mathcal{O}_{F_v}^\times}) = \chi^{-1}(U_1 \cap \mu_\infty)$ is an open compact subgroup, so it has finite index by compactness of $\prod_{v\nmid p} \mathcal{O}_{F_v}^\times$. Thus for all but finitely many places, the restriction of χ is trivial (more generally any automorphic representation is unramified almost everywhere, see Flath's theorem mentioned above). By putting the behaviour at $v \mid \infty$, $v \mid p$, $v \nmid p, \infty$ we see that $\tilde{\chi}$ is locally constant with open kernel $\mathcal{U}$. Since $\mathcal{U}$ contains $(F_\infty)^{\times,\circ}$, we see that the double coset $F^\times \setminus \mathbb{A}_F^\times / \mathcal{U}$ is finite since replacing $\mathcal{U}$ by $(F_\infty)^{\times,\circ}$ the double coset is compact. That means there exists finitely many g_i such that the value of $\tilde{\chi}$ is determined by its restriction to $F^\times$,$\mathcal{U}, g_i$, this implies that the image of the character lies in some number field $E \subset \overline{\mathbb{Q}}$.

This means that the Hecke character χ differs from a character $\tilde{\chi}$ taking values in number field by a very simple algebraic character. Without algebraicity, automorphic representations naturally form families in real or complex topology, e.g. twisting by $||^s$, on the other hand p-adic Galois characters form families in p-adic topology. In order to state Langlands reciprocity, we need to either impose such algebraicity condition or introduce more general objects on both sides.

If F is a number field, then for each place v of F, recall $\mathcal{O}_{F_v}^\times = \{x \in F_v : |x|_v = 1\}$, and $\mathfrak{m}_{F_v} = \{x \in F_v : |x|_v < 1\}$. Define $\mathcal{O}_{\overline{F_v}}$ and $\mathfrak{m}_{\overline{F_v}}$ similarly, and $\overline{k_v} = \mathcal{O}_{\overline{F_v}} / \mathfrak{m}_{\overline{F_v}}$. The difference is that this is not discretely valued and also $\mathfrak{m}_{\overline{F_v}}^2 = \mathfrak{m}$.

For a (not necessarily finite) extension E/F, we say it is unramified at $w \mid v$ if I_{F_v} has image 1 in $Gal(E/F)$. More generally, a continuous homomorphism $\rho : G_F \to H$ where H is a topological group is unramified at $w \mid v$ if $\rho(I_{F_v}) = 1$, i.e. $\rho(Frob_v)$ is defined (depending on the emdedding of the local Galois group into the global Galois group, but the conjugacy class of $\rho(Frob_v)$ is well-defined).

For any subset of places P of F, We say P has density δ if

$$\lim_{N \to \infty} \frac{|\{v \in P : \#k_v \le N\}|}{|\{v : \#k_v \le N\}|} = \delta$$

. By Prime Number theorem, the denominator is $\sim N / \log N$.

Recall Cebotarev density theorem, if X is a union of conjugacy classes in $H = Gal(E/F)$, the set of places whose Frobenius lies in X has density $|X|/|H|$. The first corollary is that each $h \in Gal(E/F)$ is the Frobenius elements of infinitely many unramified places of E. The second corollary is the for E/F Galois but not necessarily finite, Frobenius elements of unramified places of E are dense in $Gal(E/F)$.

If k is a characteristic zero field, then for any ρ_1, ρ_2 two irreducible semisimple (direct sum of irreducible) finite-dimensional representations of k-algebra Λ with $tr(\rho_1) = tr(\rho_2)$, then $\rho_1 \cong \rho_2$ (Bourbaki, Ch 8, chapter 12, section 1, prop. 3). Combined with Cebotarev density theorem, we get that if ρ_1, ρ_2 are continuous semisimple representations such that both are unramified outside a given finite subset of places, then $\rho_1 \cong \rho_2$ iff $char(\rho_1(Frob_v)) = char(\rho_2(Frob_v))$.

We say E/F_v is tamely ramified if $p \nmid e$. There is a maximal unramified (resp. tamely ramified) extension F^{nr} and F^{tr} of F_v in $\overline{F_v}$. If we let ϖ be any uniformizer of F_v, then $F_v^{nr} = \bigcup_{p \nmid m} F_v(\mu_m)$. Similarly, $F_v^{tr} = \bigcup_{p \nmid m} F_v^{nr}(\varpi^{1/m})$. Let $P_{\overline{F_v}}$ be the wild inertia (whose retriction to F_v^{tr} is trivial), it is a pro-p-group, and the quotient of I_{F_v} is $Gal(F_v^{tr}/F_v^{ur}) \xrightarrow[\cong]{t} (\overline{k_v})^\times \cong \prod_{p|\ell} \mathbb{Z}_\ell(1)$ (the identification is via the Kummer map). Under this identification, we have $t(Frob_v \sigma Frob_v^{-1}) = |k_v|^{-1} t(\sigma)$ (by considering the action of geometric Frobenius on roots of unity)

Reference:

For topology on adelic point of algebraic groups: Brian conrad's paper https://math.stanford.edu/~conrad/papers/adelictop.pdf

For restriction of scalar: appendix of Conrad-Gabber-Prasad