automorphy lifting lecture 7-9

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The space of cuspidal automorphic forms $\mathcal{A}_0(GL_n(F) \setminus GL_n(\mathbb{A}_F))$ is the set of all (complex-valued) functions $\varphi : GL_n(F) \setminus GL_n(\mathbb{A}_F) \to \mathbb{C}$ such that

- φ is smooth (i.e. locally constant in the finite places and smooth in the infinite places);
- φ is K-finite/admissible (The space of right translates under right translates by product of maximal compact subgroups (speficied as follows) is finite-dimensional. For the finite places, we use $GL_n(\widehat{\mathcal{O}_F})$ where $\widehat{\mathcal{O}_F}$ is the profinite completion of $\mathcal{O}_F$, so it is $\prod_v \mathcal{O}_{F,v}$. For the infinite places, we use $U_\infty = \prod_{v|\infty} U_v$ where U_v is a maximal compact subgroup of $GL_n(F_v)$, required to be $O(n)$ when v is real and $U(n)$ when v is complex.);
- φ is $\mathfrak{h}$-finite (should be treated together with the previous condition at infinite place; here $\mathfrak{h}$ is the center of the universal envelopping algebra $\mathcal{U}(\mathfrak{g})$ where $\mathfrak{g} := Lie(GL_n(F_\infty)) \otimes \mathbb{C}$ and the action is by $(X\varphi)(g) = \frac{d}{dt}|_{t=0}(\varphi(g\exp(tX)))$ and extended to $\mathcal{U}(\mathfrak{g})$ by universal property, c.f. the Harish-Chandra isomorphism);
- φ is slowly increasing (polynomial growth);
- φ is cuspidal (integral of the left-translates $\varphi(ug)$ along every unipotent radical of the standard parabolic subgroup vanishes);

The space $\mathcal{A}_0(GL_n(F) \setminus GL_n(\mathbb{A}_F))$ is not quite a representation of $GL_n(\mathbb{A}_F)$, because U_∞-finite is not preserved under right translation by g (instead it is $gU_\infty g^{-1}$-finite; note that there is no problem at finite places). However it does admits an action by $GL_n(\mathbb{A}_F^\infty) \times U_\infty$ and an action by $\mathfrak{g}$, and they are related by

$$g(X\varphi) = (ad(g_\infty)X)(g\varphi).$$

Another remark is that in the non-Archimedean case requring K-finite is the same as admissibility (the space of fixed vectors of any compact open subgroup is finite-dimensional), and the latter is more convenient since we don't need to keep track of isotypic components (see Getz, Intro to Automorphic representations, Prop. 5.3.11).

A third remark is that automorphic representations are factorizable, i.e. an irreducible $\pi = \otimes'_v \pi_v$ (Flath's theorem, see Theorem 5.7.1 for a proof).

The center $\mathfrak{h} = \otimes_{\tau:F\to\mathbb{C}}\mathfrak{h}_\tau$ of the universal envelopping algebra at infinite places act by π by scalars. By the Harish-Chandra isomorphism, we have $\mathfrak{h}_\tau \cong \mathbb{C}[x_1, ...x_n]^{S_n}$, so each τ gives us n complex numbers. We make the following definition: If $HC_\tau \subseteq \mathbb{Z}$ for each τ, then it is algebraic. If it has n distinct elements for each τ, then it is regular. The regular algebraic representations are accessible via topology since they appear in the Betti cohomology of the symmetric space $GL_n(F) \setminus GL_n(\mathbb{A}_F)/K$ for some choice of compact subgroup K.

The case $n = 1$ of Global Langlands is a reformulation of class field theory. A cuspidal automorphic representation of GL_1 is just a continuous character $\chi : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times$. The algebraicity condition says that $\chi|_{(F_\infty^\times)^\circ}$ looks like $x \mapsto \prod_{\tau:F\to\mathbb{C}} \tau(x)^{-n_\tau}$ for some integers n_τ. From χ we would like to produce a Galois representation. First we define $\tilde{\chi} : \mathbb{A}_F^\times \to \mathbb{C}^\times$ by $\tilde{\chi} := \chi(x)\prod_{\tau:F\to\mathbb{C}} \tau(x)^{n_\tau}$ (note this no longer trivial on $F^\times$, but it takes $F^\times$ to $\overline{\mathbb{Q}}^\times$). By continuity of the character, it is invariant by some open compact subgroup $U \subset \mathbb{A}_F^\times$ and also on $(F_\infty^\times)^\circ$ by construction. A fundamental fact is that any quotient $\mathbb{A}_F^\times/F^\times U(F_\infty^\times)^\circ$ is finite, so $\tilde{\chi}$ is valued in $\overline{\mathbb{Q}}^\times$ on the entire $\mathbb{A}_F^\times$.

We can now use the isomorphism $\eta : \mathbb{C} \to \overline{\mathbb{Q}_p}$ (restricted to $\overline{\mathbb{Q}}^\times$) to make $\tilde{\chi}$ valued in $\overline{\mathbb{Q}_p}$ and then modify it at the places above p to make it invariant by $F^\times$ by undoing the integral twist:

$$\chi^{(p)} = \eta \circ \tilde{\chi}(x) \prod_{\tau:F\to\overline{\mathbb{Q}_p}} \tau(x_p)^{-n_{\eta^{-1}\circ\tau}}$$

Since this involves places above p, the character $\chi^{(p)}$ will factor through $\mathbb{A}_F^\times/\overline{F^\times(F_\infty^\times)^\circ} \xrightarrow[Art_F]{\cong} G_F^{ab}$.

Similarly, starting from an algebraic p-adic Hecke character, we can get a complex valued algebraic Hecke character. One thing to note is that the image of $\tilde{\chi}$ lies in a number field. First, the image of $\mathbb{A}_F^\times$ under $\prod_{\tau:F\to\overline{\mathbb{Q}_p}} \tau(x)^{n_\tau}$ lies in F^{Gal}, the Galois closure of F (the image of F^{Gal} is independent of choice of the embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}$). At other places the image of $\tilde{\chi}$ is unchanged. For the infinite place, we must have $\chi|_{(F_\infty)^{\times,\circ}} = 1$ since the target is totally disconnected. For finite places $v \nmid p$, the incompatibility of the profinite topologies implies that there is an open neighborhood of 1 such that χ is trivial. Hence by compactness of $\mathcal{O}_{F_v}^\times$, the image $\chi(\mathcal{O}_{F_v}^\times)$ is finite, hence it has image in roots of unity μ_∞. Since $U_1 \cap \mu_\infty = \{1\}$ where U_1 is the open subgroup $1 + \varpi\mathcal{O}_{\overline{\mathbb{Q}_p}}$. Thus $ker(\chi|_{\prod_{v\nmid p}\mathcal{O}_{F_v}^\times}) = \chi^{-1}(U_1 \cap \mu_\infty)$ is an open compact subgroup, so it has finite index by compactness of $\prod_{v\nmid p}\mathcal{O}_{F_v}^\times$. Thus for all but finitely many places, the restriction of χ is trivial (more generally any automorphic representation is unramified almost everywhere, see Flath's theorem mentioned above). By putting the behaviour at $v \mid \infty$, $v \mid p$, $v \nmid p, \infty$ we see that $\tilde{\chi}$ is locally constant with open kernel $\mathcal{U}$. Since $\mathcal{U}$ contains $(F_\infty)^{\times,\circ}$, we see that the double coset $F^\times \setminus \mathbb{A}_F^\times/\mathcal{U}$ is finite since replacing $\mathcal{U}$ by $(F_\infty)^{\times,\circ}$ the double coset is compact. That means there exists finitely many g_i such that the value of $\tilde{\chi}$ is determined by its restriction to $F^\times$,$\mathcal{U}, g_i$, this implies that the image of the character lies in some number field $E \subset \overline{\mathbb{Q}}$.

This means that the Hecke character χ differs from a character $\tilde{\chi}$ taking values in number field by a very simple algebraic character. Without algebraicity, automorphic representations naturally form families in real or complex topology, e.g. twisting by $||^s$, on the other hand p-adic Galois characters form families in p-adic topology. In order to state Langlands reciprocity, we need to either impose such algebraicity condition or introduce more general objects on both sides.

If F is a number field, then for each place v of F, recall $\mathcal{O}_{F_v}^\times = \{x \in F_v : |x|_v = 1\}$, and $\mathfrak{m}_{F_v} = \{x \in F_v : |x|_v < 1\}$. Define $\mathcal{O}_{\overline{F_v}}$ and $\mathfrak{m}_{\overline{F_v}}$ similarly, and $\overline{k_v} = \mathcal{O}_{\overline{F_v}}/\mathfrak{m}_{\overline{F_v}}$. The difference is that this is not discretely valued and also $\mathfrak{m}_{\overline{F_v}}^2 = \mathfrak{m}$.

For a (not necessarily finite) extension E/F, we say it is unramified at $w \mid v$ if I_{F_v} has image 1 in $Gal(E/F)$. More generally, a continuous homomorphism $\rho : G_F \to H$ where H is a topological group is unramified at $w \mid v$ if $\rho(I_{F_v}) = 1$, i.e. $\rho(Frob_v)$ is defined (depending on the emdedding of the local Galois group into the global Galois group, but the conjugacy class of $\rho(Frob_v)$ is well-defined).

For any subset of places P of F, We say P has density δ if

$$\lim_{N\to\infty} \frac{|\{v \in P : \#k_v \le N\}|}{|\{v : \#k_v \le N\}|} = \delta$$

. By Prime Number theorem, the denominator is $\sim N/\log N$.

Recall Cebotarev density theorem, if X is a union of conjugacy classes in $H = Gal(E/F)$, the set of places whose Frobenius lies in X has density $|X|/|H|$. The first corollary is that each $h \in Gal(E/F)$ is the Frobenius elements of infinitely many unramified places of E. The second corollary is the for E/F Galois but not necessarily finite, Frobenius elements of unramified places of E are dense in $Gal(E/F)$.

If k is a characteristic zero field, then for any ρ_1, ρ_2 two irreducible semisimple (direct sum of irreducible) finite-dimensional representations of k-algebra Λ with $tr(\rho_1) = tr(\rho_2)$, then $\rho_1 \cong \rho_2$ (Bourbaki, Ch 8, chapter 12, section 1, prop. 3). Combined with Cebotarev density theorem, we get that if ρ_1, ρ_2 are continuous semisimple representations such that both are unramified outside a given finite subset of places, then $\rho_1 \cong \rho_2$ iff $char(\rho_1(Frob_v)) = char(\rho_2(Frob_v))$.

We say E/F_v is tamely ramified if $p \nmid e$. There is a maximal unramified (resp. tamely ramified) extension F^{nr} and F^{tr} of F_v in $\overline{F_v}$. If we let ϖ be any uniformizer of F_v, then $F_v^{nr} = \bigcup_{p\nmid m} F_v(\mu_m)$. Similarly, $F_v^{tr} = \bigcup_{p\nmid m} F_v^{nr}(\varpi^{1/m})$. Let $P_{\overline{F_v}}$ be the wild inertia (whose retriction to F_v^{tr} is trivial), it is a pro-p-group, and the quotient of $\overline{I_{F_v}}$ is $Gal(F_v^{tr}/F_v^{ur}) \xrightarrow[\cong]{t} (\overline{k_v})^\times \cong \prod_{p\nmid \ell} \mathbb{Z}_\ell(1)$ (the identification is via the Kummer map). Under this identification, we have $t(Frob_v \sigma Frob_v^{-1}) = |k_v|^{-1} t(\sigma)$ (by considering the action of geometric Frobenius on roots of unity).

Let $t_\ell : I_{F_v} \to \mathbb{Z}_\ell(1)$ denote the projection to the ℓ-th place. To compare representations with coefficient in different characteristic, we need to remove the influence of topology. If we are considering ℓ-adic representations of the local Galois group at p and $p \neq \ell$, then the wild inertia will have finite image since the characteristic doesn't match. But the tame inertia (projection to the ℓ-th place) could have infinite image. Grothendieck's ℓ-adic monodromy theorem allows us to describe what this image look like and leads to the notion of Weil-Deligne representation.

> Given continuous representation $r : G_{F_v} \to GL(V)$ or $r : W_{F_v} \to GL(V)$ where $V = \overline{\mathbb{Q}_\ell}^n$ for some n. Then there exists open subgroup $I_0 \subset I_{F_v}$ (of finite index) such that $r(\sigma)$ is unipotent for all $\sigma \in I_0$.

Let $H = G_{F_v}$ or W_{F_v}. As explained before there exists $E/\mathbb{Q}_\ell$ finite extension and a finite index subgroup $I_0 \subset I_{F_v}$ such that $r(I_0) \subset 1 + \ell^2 M_n(\mathcal{O}_E) \subset GL_n(\mathcal{O}_E)$. Since $1 + \ell M_n(\mathcal{O}_E)$ is a pro-ℓ group, $r|_{I_{F_v}}$ factors through t_ℓ. Since $r(\sigma)$ depends only on $t_\ell(\sigma)$, and $r(\sigma)$ is conjugate to $r(Frob_v^a \sigma Frob_v^{-a}) = r(\sigma^{\chi_\ell(a)})$ for $a \in \mathbb{Z}$. Let $X := \log_\ell r(\sigma)$ (which converges by the choice of I_0). Then X is conjugate to $\log_p(r(\sigma^{\chi_\ell(a)})) = \chi_\ell(a)X$. If $a_i(X)$ is the i-th coefficeint of the characteristic polynomial, then $a_i(X) = \chi_\ell(a)^i a_i(X)$. If for some $0 \leq i \leq n-1$ we have $\chi_\ell(a)^i = 1$ for all $a \in \mathbb{Z}$, then χ_ℓ has finite order, but this contradict the key assumption that k_v satisfies. Hence we must have for $a_i(X) = 0$ for every $0 \leq i \leq n-1$. Thus the characteristic polynomial is just $X^n = 0$, i.e. X is nilpotent.

Thus we can write $r(\sigma) = \exp(\overline{r}(t_\ell(\sigma)))$ for some nilpotent representation $\overline{r} : \mathbb{Q}_\ell(1) \to End(V)$, i.e. a nilpotent matrix $N_0 : V(1) \to V$ such that for any lift $\phi \in W_{F_v}$ of $Frob_v$ we have $r(\phi) N_0 r(\phi)^{-1} = |k_v|^{-1} N_0$.

Recipe: Choose any lift $\phi \in W_{F_v}$ of $Frob_v$ and $\tau : \mathbb{Z}_\ell \to \mathbb{Z}_\ell(1)$ (i.e. a system of ℓ^n-th root of unity) inducing $\tau_v : V \to V(1)$. Define $\rho(\phi^a \sigma) = r(\phi^a \sigma) \exp(-N_0 t_\ell(\sigma))$ and $N := N_0 \circ \tau_v \in End(V)$. Such a pair (ρ, N) is a Weil-Deligne representation. The formal definition is as follows:

> Let E be any field. A Weil-Deligne representation of F_v is a pair (ρ, N) where $\rho : W_{F_v} \to GL(V)$ is a continuous representation where $V = E^n$ (equipped with the discrete topology) and $N \in End_E(V)$ nilpotent such that for all lifts $\phi \in W_{F_v}$ of $Frob_v$, we have $\rho(\phi)N\rho(\phi)^{-1} = |k_v|^{-1}N$.

(Deligne) There is an equivalence of categories:

$$\{\text{continuous representation of } r : W_{F_v} \to GL_n(\overline{\mathbb{Q}_\ell})\} \to \{n\text{−dimensional WD-}$$

Intuition: Think of F_v as a punctured disk with k_v the puncture. The Galois group G_{F_v} is the fundamental group. The puncture has fundamental group generated by the Frobenius. The kernel I_{F_v} control the monodromy. The key assumption satisfied by k_v is that no finite extension of k_v contains all ℓ^n-th power of unity.

A WD-representation (ρ, N) is unramified if $N = 0$ and $\rho(I_v) = 1$. We say (ρ, N) is Frobenius-semisimple if ρ is simisimple. Any (ρ, N) has a Frob-semisimplification $(\rho, N)^{ss}$: Pick any lift $\phi \in W_{F_v}$ of $Frob_v$, and write $\rho(\phi) = \rho(\phi)_{ss}\rho(\phi)_u$. Then we define $(\rho, N)^{ss} = (\rho', N)$ with $\rho'(\phi^a\sigma) = \rho(\phi)^a_{ss}\rho(\sigma)$ (so we only semisimplify the action of the Frobenius, but $\rho(\sigma)$ has finite order because of continuity hence also semisimple; it is useful since most of the time we only know the characteristic polynomial).

Note that we use the fact that the coefficient field has characteristic zero. Otherwise we need to impose semisimplicity assumption on $\rho|_{I_{F_v}}$.

Reference:

For topology on adelic point of algebraic groups: Brian conrad's paper https://math.stanford.edu/~conrad/papers/adelictop.pdf

For restriction of scalar: appendix of Conrad-Gabber-Prasad