Deligne-Lusztig theory algebraic group preliminaries

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Let $G_u \subset G$ be the set of unipotent elements. This is a closed subset and hence an algebraic (affine) variety. Note that G acts on G_u by conjugation, the orbits of which are called unipotent orbits. We will see later that if G is reductive then there are finitely many unipotent orbits. In the case of $\mathbb{C}$, it is due to Dynkin-Kostant. In the case of $k = \overline{\mathbb{F}_q}$, it is proved by Richardson (classical groups) and Lusztig (exceptional groups). This is one of Lusztig's motivation, that is to give a uniform proof of finiteness of unipotent orbits using Deligne-Lusztig theory.

In the case of GL_2, nilpotent matrices are those with determinant and trace zero. Thus G_u can be identified with $\{(x,y,z) : x^2 + yz = 0\}$, sending $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $(1-a, b, c)$, which looks like a cone with singularity at origin. There are two unipotent orbits: the identity and the conjugacy class of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. In GL_n over the complex numbers, the number of conjugacy classes is the number of partitions of n by Jordan normal form.

Every elements of torus is semisimple. One proof over $\overline{\mathbb{F}_q}$ is use the criterion that an element is semisimple iff its order is coprime to p and unipotent if its order is a power of q. A characterization of torus is that T is a torus iff T is connected commutative algebraic group consisting of semi-simple elements. The idea is that we can choose a closed embedding of T into the diagonal torus, and it remains to show that closed connected subgroups of the diagonal torus is of the form $\mathbb{G}_m^l$ for some l.

Let T be a torus. The Weyl group $W_T := N(T)/Z(T)$ is finite (proof is highly nontrivial).

Below are some facts from the theory of algebraic groups:

Irreducible iff connected iff geometrically irreducible iff geometrically connected;

geometrically reduced iff smooth (This is because of generic smoothness and homogeneity); But reduced need not imply geometrically reduced over nonperfect fields. The identity component is geometrically connected (since it contains a rational point, see here, Lemma 33.7.14 for a proof.)

Let G be an algebraic group. There exists a maximal torus $T \subset G$ (with respect to inclusion).

The following is an important theorem: Any maximal torus in G are all conjugate to each other. In the case of compact Lie groups, this follows from Lefschetz fixed-point formula due to Hermann Weyl, in the case of algebraic groups, this follows from Borel fixed-point formula.

Assume G is defined over $\mathbb{F}_q$. A maximal torus of G^F is a subgroup of the form T^F where $T \subset G$ is a maximal torus stable under F. For any $g \in G^F$, then $gT^Fg^{-1} = (gTg^{-1})^F \subset G^F$ is again a maximal torus.

The natural question is to classify maximal torus in G^F up to G^F-conjugacy. For GL_2 the diagonal maximal torus T_2 is F-stable. Let T be a F-stable torus, then we can write $T = gT_2g^{-1}$, then we can check that $n := g^{-1}F(g) \in N(T_2)$ for T to be F-stable. Since $N(T_2) = T_2 \sqcup T_2 s$, where s is antidiagonal matrix with all ones. Thus T^F is isomorphic to $\{t \in T_2 : nF(t)n^{-1} = t\}$ by sending $x = gtg^{-1}$ to t (the condition that x is fixed by Frobenius means that $F(g)F(t)F(g^{-1}) = gtg^{-1}$). If $g \in G^F$ (or more generally $n \in T_2$) then this is just T_2^F but we get something new; but if $n \in T_2 s$, then $T^F \cong \left\{ s \begin{pmatrix} a^q & \\ & b^q \end{pmatrix} s^{-1} = \begin{pmatrix} a & \\ & b \end{pmatrix} \right\} \cong \left\{ \begin{pmatrix} a & \\ & a^q \end{pmatrix} : a \in \mathbb{F}_{q^2}^\times \right\}$. So we get two G^F-conjugacy classes of T^F, but we need to ensure that we can find $g \in GL_2$ such that $n \in T_2 s$. This is eassy, e.g. take $g = \begin{pmatrix} x & x^q \\ 1 & 1 \end{pmatrix} \notin GL_2^F$, then $g^{-1}F(g) = s$.

From the above discussion we also see that it is important to consider the function $g^{-1}F(g)$ and the level set $\{g : g^{-1}F(g) = w\}$ for $w \in W$ where W is the Weyl group $N(T)/T$. This is precisely the Deligne-Lusztig variety $X(w)$.

Some preliminaries on reductive groups:

(Kolchin's theorem) If $U \subset GL_n$ is a unipotent group, then there exists $g \in GL_n$ such that $gUg^{-1} \subset U_n$ the standard unipotent group. Equivalently, there exists a complete flag fixed by U and $U|_{V_{i+1}/V_i} = id$.

(Lie-Kolchin's theorem) Let $B \subset GL_n$ be a connected solvable group. Then there exists $g \in GL_n$ such that $gBg^{-1} \subset B_n$ the upper triangular Borel subgroup. Equivalently, there exists a complete flag fixed by B.

The first theorem is equivalent to the assertion that any representation of a unipotent group has a fixed vector. Similarly, the second theorem is equivalent to the assertion that any represention of a connected solvable group has a fixed line.

Another remark is that although Kolchin's theorem holds true for any field k but Lie-Kolchin does not, e.g. the standard representation $SO_2(\mathbb{R})$ has no fixed lines.

Kolchin's theorem also implies that unipotent groups are nilpotent as abstract group. The following gives a structure theorem of connected solvable groups:

Let G be a connected solvable group, then G_u is a connected normal subgroup of G and $G \cong G_u \rtimes T$ where T is a maximal torus. Moreover, $W = N(T)/T = \{1\}$.

As a corollary, we have $G/G_u \cong T$. It is actually remarkable that the quotient has an algebro-geometric structure.

Now we get to the notion of reductive groups. Let G eb a connected algebraic group. The radical $R(G)$ is the maximal connected, solvable, normal subgroup of G. The unipotent part $R(G)_u$ is called the unipotent radical. G is called reductive if $R(G)_u = e$ and G is called semisimple if $R(G) = e$.

If $char k = 0$, then G is reductive iff any representation of G is a direct sum of irreducibles. All the classical groups GL_n, Sp_{2n} and SO_n are reductive. The proof in the case of GL_n consists of showing $V = (k^n)^{R_u}$ is equal to k^n where GL_n acts on k_n via the standard representation. By Kolchin's theorem we can find $0 \neq v \in V$. but since R_u is normal, V is acted on by GL_n and so $V = k^n$ by transitivity of the action. For other classical groups use the same trick.

Reductive essentially means the group is an extension of torus by semisimple groups (and also the finite component group if we consider discconected reductive groups.)

Borel subgroups: Let G be a conneccted algebraic group. A subgroup B is Borel if B is maximal among all connected solvable subgroups (no normality). $B_n \subset GL_n$ is Borel (Exercise). By Lie-Kolchin theorem, any Borel subgroup B is of the form gB_ng^{-1} for some $g \in GL_n$. Also, $N(B_n) = B_n$ (Exercise). There is a set bijection between GL_n/B_n to the set of Borel subgroups of GL_n by sending gB_n to $B = gB_ng^{-1}$. On the other hand, there is a geometric interpretation of GL_n/B_n as the set of complete flags in k^n, and this turns out to have the structure of a projective variety.

For a split torus, the character group $X^*(T)$ is defined to be $Hom_k(T, \mathbb{G}_m)$ and the cocharacter group $X_*(T)$ is $Hom_k(\mathbb{G}_m, T)$. Both are free $\mathbb{Z}$-modules. In the nonsplit case we define $X^*(T)$ to be $X^*(T_{\overline{k}})$, which is equipped with a discrete action of $Gal(\overline{k}/k)$ (via action on both domain and codomain, see this answer for an example. This turns out to be an equivaluence between category of k-tori and category of discrete $\mathbb{Z}[Gal(\overline{k}/k)]$-modules.

Sketch why an affine algebraic group admits a closed immersion into some GL_n: An action of an affine algebraic group on a vector space (possibly infinite-dimensional) can be defined as a natural transformation $G \to \underline{\mathrm{Aut}}(V)$, so it is the same as a functorial action of $G(R')$ acting on $V_{R'}$ for each R-algebra R'. The idea is that G acts on the coordinate ring $k[G]$, but $K[G]$ is infinite-dimensional. However, it turns out that the action is via a k-algebra automorphism, so we are done if we can show that the finite number of generators is contained in a finite-dimensional G-stable subspace. This turns out to reduced to the associativity axiom of a group action, for details see this handout, section 12.2.

Key lemma in showing Jordan decomposition for arbitrary algebraic groups: If G is given as a closed subgroup of GL_n, how do we single out the elements of G? For classical groups they are defined as groups of matrices that preserve a certain bilinear or sesquilinear form. Thus we would like to similarly realize an arbitrary algebraic group G as stablizer of a certain action. It turns out this is indeed possible (see here Theorem 14.1.1; the idea is that G stablizes the kernel $k[GL_n] \to k[G]$). Using this we can reduced showing the existence of Jordan decomposition of an arbitrary affine algebraic group to showing a closed immersion $GL_n \to GL_m$ preserves semisimple and unipotent part, for details see here.

Proof of Kolchin's theorem: The idea is to reduced to the case of algebraically closed fields, and use Wedderburn's lemma:

> If $k = \overline{k}$ and $\Gamma \subset GL(V)$ is such that V is an irreducible Γ-representation, then $End_k(V)$ is generated as a k-algebra by Γ.

Using this lemma, we could show any irreducible representation ρ of a unipotent group G is trivial. The idea is to write $\rho(g) = 1 + x$, where x is nilpotent, and unipotence of G implies that $tr(x\rho(g')) = 0$ for any $g' \in G$. Then by Wedderburn's lemma $tr(xy) = 0$ for any $y \in End_k(V)$, and hence $x = 0$ by the nondegeneracy of the trace pairing.

Reference: https://virtualmath1.stanford.edu/~conrad/252Page/handouts/alggroups.pdf