Deligne-Lusztig theory algebraic group preliminaries

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Let $G_u \subset G$ be the set of unipotent elements. This is a closed subset and hence an algebraic (affine) variety. Note that G acts on G_u by conjugation, the orbits of which are called unipotent orbits. We will see later that if G is reductive then there are finitely many unipotent orbits. In the case of $\mathbb{C}$, it is due to Dynkin-Kostant. In the case of $k = \overline{\mathbb{F}_q}$, it is proved by Richardson (classical groups) and Lusztig (exceptional groups). This is one of Lusztig's motivation, that is to give a uniform proof of finiteness of unipotent orbits using Deligne-Lusztig theory.

In the case of GL_2, nilpotent matrices are those with determinant and trace zero. Thus G_u can be identified with $\{(x, y, z) : x^2 + yz = 0\}$, sending $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $(1 - a, b, c)$, which looks like a cone with singularity at origin. There are two unipotent orbits: the identity and the conjugacy class of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. In GL_n over the complex numbers, the number of conjugacy classes is the number of partitions of n by Jordan normal form.

Every elements of torus is semisimple. One proof over $\overline{\mathbb{F}_q}$ is use the criterion that an element is semisimple iff its order is coprime to p and unipotent if its order is a power of q. A characterization of torus is that T is a torus iff T is connected commutative algebraic group consisting of semi-simple elements. The idea is that we can choose a closed embedding of T into the diagonal torus, and it remains to show that closed connected subgroups of the diagonal torus is of the form $\mathbb{G}_m^l$ for some l.

Let T be a torus. The Weyl group $W_T := N(T)/Z(T)$ is finite (proof is highly nontrivial).

Below are some facts from the theory of algebraic groups:

Irreducible iff connected iff geometrically irreducible iff geometrically connected;

geometrically reduced iff smooth (This is because of generic smoothness and homogeneity); But reduced need not imply geometrically reduced over nonperfect fields. The identity component is geometrically connected (since it contains a rational point, see here, Lemma 33.7.14 for a proof.)

Let G be an algebraic group. There exists a maximal torus $T \subset G$ (with respect to inclusion).

The following is an important theorem: Any maximal torus in G are all conjugate to each other. In the case of compact Lie groups, this follows from Lefschetz fixed-point formula due to Hermann Weyl, in the case of algebraic groups, this follows from Borel fixed-point formula.

Assume G is defined over $\mathbb{F}_q$. A maximal torus of G^F is a subgroup of the form T^F where $T \subset G$ is a maximal torus stable under F. For any $g \in G^F$, then $gT^Fg^{-1} = (gTg^{-1})^F \subset G^F$ is again a maximal torus.

The natural question is to classify maximal torus in G^F up to G^F-conjugacy. For GL_2 the diagonal maximal torus T_2 is F-stable. Let T be a F-stable torus, then we can write $T = gT_2g^{-1}$, then we can check that $n := g^{-1}F(g) \in N(T_2)$ for T to be F-stable. Since $N(T_2) = T_2 \sqcup T_2 s$, where s is antidiagonal matrix with all ones. Thus T^F is isomorphic to $\{t \in T_2 : nF(t)n^{-1} = t\}$ by sending $x = gtg^{-1}$ to t (the condition that x is fixed by Frobenius means that $F(g)F(t)F(g^{-1}) = gtg^{-1}$). If $g \in G^F$ (or more generally $n \in T_2$) then this is just T_2^F but we get something new; but if $n \in T_2 s$, then $T^F \cong \left\{ s \begin{pmatrix} a^q & \\ & b^q \end{pmatrix} s^{-1} = \begin{pmatrix} a & \\ & b \end{pmatrix} \right\} \cong \left\{ \begin{pmatrix} a & \\ & a^q \end{pmatrix} : a \in \mathbb{F}_{q^2}^\times \right\}$. So we get two G^F-conjugacy classes of T^F, but we need to ensure that we can find $g \in GL_2$ such that $n \in T_2 s$. This is eassy, e.g. take $g = \begin{pmatrix} x & x^q \\ 1 & 1 \end{pmatrix} \notin GL_2^F$, then $g^{-1}F(g) = s$.

From the above discussion we also see that it is important to consider the function $g^{-1}F(g)$ and the level set $\{g : g^{-1}F(g) = w\}$ for $w \in W$ where W is the Weyl group $N(T)/T$. This is precisely the Deligne-Lusztig variety $X(w)$.

Some preliminaries on reductive groups:

(Kolchin's theorem) If $U \subset GL_n$ is a unipotent group, then there exists $g \in GL_n$ such that $gUg^{-1} \subset U_n$ the standard unipotent group. Equivalently, there exists a complete flag fixed by U and $U|_{V_{i+1}/V_i} = id$.

(Lie-Kolchin's theorem) Let $B \subset GL_n$ be a connected solvable group. Then there exists $g \in GL_n$ such that $gBg^{-1} \subset B_n$ the upper triangular Borel subgroup. Equivalently, there exists a complete flag fixed by B. (connectedness is crucial, e.g. S_3)

The first theorem is equivalent to the assertion that any representation of a unipotent group has a fixed vector. Similarly, the second theorem is equivalent to the assertion that any represention of a connected solvable group has a fixed line.

Another remark is that although Kolchin's theorem holds true for any field k but Lie-Kolchin does not, e.g. the standard representation $SO_2(\mathbb{R})$ has no fixed lines.

Kolchin's theorem also implies that unipotent groups are nilpotent as abstract group. The following gives a structure theorem of connected solvable groups:

Let G be a connected solvable group, then G_u is a connected normal subgroup of G and $G \cong G_u \rtimes T$ where T is a maximal torus. Moreover, $W = N(T)/T = \{1\}$.

As a corollary, we have $G/G_u \cong T$. It is actually remarkable that the quotient has an algebro-geometric structure.

Now we get to the notion of reductive groups. Let G eb a connected algebraic group. The radical $R(G)$ is the maximal connected, solvable, normal subgroup of G. The unipotent part $R(G)_u$ is called the unipotent radical. G is called reductive if $R(G)_u = e$ and G is called semisimple if $R(G) = e$.

If $chark = 0$, then G is reductive iff any representation of G is a direct sum of irreducibles. All the classical groups GL_n, Sp_{2n} and SO_n are reductive. The proof in the case of GL_n consists of showing $V = (k^n)^{R_u}$ is equal to k^n where GL_n acts on k_n via the standard representation. By Kolchin's theorem we can find $0 \neq v \in V$. but since R_u is normal, V is acted on by GL_n and so $V = k^n$ by transitivity of the action. For other classical groups use the same trick.

Reductive essentially means the group is an extension of torus by semisimple groups (and also the finite component group if we consider discconected reductive groups.)

Borel subgroups: Let G be a conneccted algebraic group. A subgroup B is Borel if B is maximal among all connected solvable subgroups (no normality). $B_n \subset GL_n$ is Borel (Exercise). By Lie-Kolchin theorem, any Borel subgroup B is of the form gB_ng^{-1} for some $g \in GL_n$. Also, $N(B_n) = B_n$ (Exercise). There is a set bijection between GL_n/B_n to the set of Borel subgroups of GL_n by sending gB_n to $B = gB_ng^{-1}$. On the other hand, there is a geometric interpretation of GL_n/B_n as the set of complete flags in k^n, and this turns out to have the structure of a projective variety.

For example, consider $G = SP_4 = \{g \in GL_4 : g^T J g = J\}$. Note that $Sp_4 \cap B_4$ is a Borel subgroup of G, by writing it as $(Sp_4 \cap T_4) \ltimes (Sp_4 \cap U_4)$. The former is

$$\begin{pmatrix} a & & & \\ & b & & \\ & & b^{-1} & \\ & & & a^{-1} \end{pmatrix}$$

and the latter is

$$\begin{pmatrix} 1 & -\alpha & \beta - \alpha\lambda & u \\ & 1 & \lambda & \beta \\ & & 1 & \alpha \\ & & & 1 \end{pmatrix}$$

. The latter can be written as product of four unipotent one-dimensional subgroup

$$\begin{pmatrix} 1 & -\alpha & & \\ & 1 & & \\ & & 1 & \alpha \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & \lambda & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \beta & \\ & 1 & & \beta \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & u \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

In terms of flags, it looks like
$0 \subset V_1 = \langle e_1 \rangle \subset V_2 = V_2^\perp = \langle e_1, e_2 \rangle \subset V_1^\perp = \langle e_1, e_2, e_3 \rangle \subset k^4$. This is an example of isotropic flags. Thus $Sp_4/(B_4 \cap Sp_4)$ can be identified with the set of isotropic flags.

To go from a Borel B of Sp_4 to an isotropic flag, by Lie-Kolchin theorem there is a line V_1 fixed by B. The orthogonal complement $V_1^\perp$ is three-dimensional and contains V_1. But then B acts on $V_1 \perp /V_1$, and there is a (non-degenerate) symplectic form that is again preserved by B. Using Lie-Kolchin theorem B has another fixed line, call the pullback V_2. The general form of isotropic flags are $0 \subset V_1 \subset V_2 \subset ...V_n = V_n^\perp \subset V_{n-1}^\perp \subset ... \subset V_1^\perp \subset k^{2n}$.

For a split torus, the character group $X^*(T)$ is defined to be $Hom_k(T, \mathbb{G}_m)$ and the cocharacter group $X_*(T)$ is $Hom_k(\mathbb{G}_m, T)$. Both are free $\mathbb{Z}$-modules. In the nonsplit case we define $X^*(T)$ to be $X^*(T_{\overline{k}})$, which is equipped with a discrete action of $Gal(\overline{k}/k)$ (via action on both domain and codomain, see this answer for an example. This turns out to be an equivaluence between category of k-tori and category of discrete $\mathbb{Z}[Gal(\overline{k}/k)]$-modules.

Sketch why an affine algebraic group admits a closed immersion into some GL_n: An action of an affine algebraic group on a vector space (possibly infinite-dimensional) can be defined as a natural transformation $G \to \underline{\mathrm{Aut}}(V)$, so it is the same as a functorial action of $G(R')$ acting on $V_{R'}$ for each R-algebra R'.

The idea is that G acts on the coordinate ring $k[G]$, but $K[G]$ is infinite-dimensional. However, it turns out that the action is via a k-algebra automorphism, so we are done if we can show that the finite number of generators is contained in a finite-dimensional G-stable subspace. This turns out to reduced to the associativity axiom of a group action, for details see this handout, section 12.2.

Key lemma in showing Jordan decomposition for arbitrary algebraic groups: If G is given as a closed subgroup of GL_n, how do we single out the elements of G? For classical groups they are defined as groups of matrices that preserve a certain bilinear or sesquilinear form. Thus we would like to similarly realize an arbitrary algebraic group G as stablizer of a certain action. It turns out this is indeed possible (see here Theorem 14.1.1; the idea is that G stablizes the kernel $k[GL_n] \to k[G]$). Using this we can reduced showing the existence of Jordan decomposition of an arbitrary affine algebraic group to showing a closed immersion $GL_n \to GL_m$ preserves semisimple and unipotent part, for details see here.

Proof of Kolchin's theorem: The idea is to reduced to the case of algebraically closed fields, and use Wedderburn's lemma:

> If $k = \overline{k}$ and $\Gamma \subset GL(V)$ is such that V is an irreducible Γ-representation, then $End_k(V)$ is generated as a k-algebra by Γ.

Using this lemma, we could show any irreducible representation ρ of a unipotent group G is trivial. The idea is to write $\rho(g) = 1 + x$, where x is nilpotent, and unipotence of G implies that $tr(x\rho(g')) = 0$ for any $g' \in G$. Then by Wedderburn's lemma $tr(xy) = 0$ for any $y \in End_k(V)$, and hence $x = 0$ by the nondegeneracy of the trace pairing.

The importance of Borel subgroups is that they cover the group. More precisely, the subgroups $gB(k)g^{-1}$ for $g \in G(k)$ cover $G(k)$. In particular, every element of $G(k)$ lies in a Borel subgroup (k algebraically closed). From this we can also derive that every semisimple element $g \in G$ lies in a torus. (There is real content to this, since g might not lie in the identity component of the Zariski closure of $\langle g \rangle$, e.g. g may have finite order, as for all unipotent elements when $char(k) > 0$.) For the proof see this handout (The idea is to use an algebro-geometric version of covering lemma, similar to the case of compact Lie groups, see this question as well, especially the proof using mapping degree).

We have the following important property of orbits of action by algebraic group:

> O_x is an open in $\overline{O_x}$.

This essentially follows from Chevalley's theorem and the homogeneity of orbits. A corollary is that if G is connected (so irreducible), then $\overline{O_x}/O_x$ is a disjoint union of orbits of strictly lower dimension. In particular, orbits of minimal dimension are closed.

A nice characterization of semisimple elements $x \in GL_n$ is that they are precisely the closed orbits in the adjoint action of GL_n (i.e. conjugacy classes).

Another corollary is that the image of a homomorphism of algebraic groups is closed, because all orbits are of the same dimension.

Borel fixed point theorem: Let B be a connected solvable group acting on a projective variety X, then B has a fixed point. The idea is to induct on $\dim B$. Since $B' = [B, B]$ is also connected solvable and has lower dimension, it has a fixed point $x \in X$. Let $X' = \cap_{b' \in B'} X^{b'}$, which is closed (by separatedness) and nonempty (since it contains x), so it is projective. We have an action of B on X' since B' is normal subgroup of B. Let $B \cdot y \subseteq X'$ be a closed orbit, so $B \cdot y \cong B/Stab_y$ (see this question for orbit-stablizer theorem in algebraic group), and this is both affine ($Stab_y$ is normal subgroup fo B since it contains B')

From this we deduce that 1. G/B is projective. 2. All Borel subgroups are conjugate. For proof of 1, let B_0 be a Borel subgroup of maximal dimension. Choose a representation $\rho : G \to GL(V)$ and a line $V_1 \subset V$ such that B_0 is the stablizer of L. Thus B_0 acts on V/V_1, and let $\mathcal{F}$ be a flag fixed by B_0 by the Lie-Kolchin's theorem and by the choice of ρ we have $B_0 = stab(\mathcal{F})$. Then $G/B \cong G \cdot \mathcal{F}$ embeds into $GL(V)/B(V)$ by the orbit stablizer theorem. It remains to show that the orbit $G \cdot \mathcal{F}$ is closed, but in fact it is a G-orbit of minimal dimension (since we choose B to have maximal dimensions among all Borels). For proof of 2, we consider an arbitrary Borel B acting on G/B_0 (which is already projective) by left multiplication, then by the Borel fixed-point theorem let xB_0 be a fixed point of B. That means $BxB_0 = xB_0 \Rightarrow x^{-1}Bx \subset B_0$. By definition of Borel this means that $x^{-1}Bx = B_0$.

Exercise: $B = N_G(B)$. This implies that $G/B \cong G/N(B) \cong \{\text{colletion of Borel of } G\}$.

As a corollary, we can show that all maximal torus $T \subset G$ are conjugate (if k is algebraically closed). For details see this handout and 24.1 of the reference.

The Cartan subgroups are connected component of centralizer of maximal torus $Z(T)^\circ$. Facts: 1. All Cartan subgroups are conjugate to each other. 2. $Z(T) = Z(T)^\circ$. 3. $Z(T)$ is nilpotent. This implies $Z(T)$ is connected and solvable, so we can find a Borel B containing T. Bruhat decomposition: The natural map

$$W = Z(T) \setminus N(T)/Z(T) \xrightarrow{\cong} B \setminus G/B$$

is a bijection. Note that $Z(T)$ is normal in $N(T)$, so W is a finite group, called the Weyl group of G. Corollary: $X = G/B = \coprod_{w \in W} X_w$ and also $Y = G/B \times G/B \cong \coprod_{w \in W} Y_w$ by noting that $G \setminus (G/B \times G/B) \cong B \setminus G/B$. The Bruhat order on Weyl group W is given by $w' \leq w$ if $X_{w'} \subseteq \overline{X_w}$. The closure $\overline{X_w} = \sqcup_{w' \leq w} X_{w'}$ are called the Schubert varieties. The advantage of using Y_w is that it is independent of the choice of Borel (it is G-orbit rather than B-orbit).

Example: If $G = GL_n$ and X be a complete flag $V_1 \subset V_2 \subset ..$ and Y a pair of flags $V_1' \subset V_2' \subset ....$ If we look at the subspace $V_1' \cap V_i$, at some point its dimension jumps from 0 to 1, denote it by $\sigma(1)$. Repeat it with V_2' and get $\sigma(2) \neq \sigma(1)$. We get a permutation $(\sigma(1), ...\sigma(n)) \in S_n$, and $Y_w = \{(V_i, V_i') : (\sigma(i)) = w\}$.

What is the closure relation: For GL_3, the dense open orbit is the one for which the full flags are $V_1 \not\subset V_2'$ and $V_1' \not\subset V_2$. There is a Bruhat order that one can read off the closure relation.

For $G = Sp_{2n}$, $W = S_n \ltimes (\pm 1)^n$ acting on T which is $t_1, ...t_n$ on the first n entries in the diagonal and $t_n^{-1}, ...t_1^{-1}$ on the last n entries. The ± 1 factors are just swapping t_i with t_i^{-1}. If $Y = V_1 \subset ...V_n = V_n^\perp$ and $V_1' \subset ...V_n' = V_n'^\perp$. If we use the above recipe and get $(\sigma(1), ...\sigma(2n))$ and $\delta = (2n + 1 - \sigma(2n), ...2n + 1 - \sigma(1))$ where δ is the permutation obtained by taking $\perp$ of V_i and V_i'. But since we have isotropic flags, we gave $\sigma = \delta$, implying $\sigma \in W$.

Exercise: Try to work out the Bruhat order for Sp_4 ($|W| = 8$).

For G reductive $Z(T) \cong T$ and the quotient $G \to G_{red} = G/R(G)_u$ identifies the Bruhat decomposition (since $R(G)_u$ is contained in every Borel B) so lots of combinatroics reduce to the reductive case.

A few remark: X_w is often called (open) Schubert variety rather than Y_w. This is because we can define a twisted version X_w^ϕ as follows: Let $\phi : G \to G$ be a surjective group homomorphism (so it induces maps on X because it sends Borel to Borel), we can consider the graph embedding $x \mapsto (x, \phi(x))$ from X to

Y. Then we define X_w^ϕ to be the inverse image of Y_w. The two most important examples are $\phi = c_g$ (conjugation by g) which gives rise to Lusztig variety and $\phi = F$ (if $k = \overline{\mathbb{F}_q}$) which gives rise to Deligne-Lusztig variety.

A character $\alpha \in X^\bullet(T)$ is called a root if there is a unipotent subgroup $U_\alpha \subset G$ (called a root subgroup) such that $tut^{-1} = \alpha(t)u$ for $t \in T$ and $u \in U$. It turns out if G is reductive, then $U_\alpha \cong \mathbb{G}_a$ and it is uniquely determined. See corollary 2.1 of the second reference. This relies on the classification of connected reductive split rank 1 group, see the last theorem of the first reference.

Construction of coroot: Let α be a root and U_α the corresponding root subgroup. Let $T_\alpha = (\ker(\alpha))^\circ$ (a codimension-one subtorus of T) and $G_\alpha = Z_G(T_\alpha)$. Note that G_α contains T and U_α, and $T_\alpha \subset Z(G_\alpha)$. The quotient G_α/T_α is a connected reductive split rank 1 group, so it is either SL_2 or PGL_2. There is a unique cocharacter $\check{\alpha} \in X_\bullet(T)$ such that $s_\alpha(\chi) := \chi - \langle \chi, \check{\alpha} \rangle \alpha$ is a reflection of $X^\bullet(T)$ and maps the set of all roots Φ to itself; moreover, we have $s_\alpha(\alpha) = -\alpha$ (i.e. $\langle \alpha, \check{\alpha} \rangle = 2$.

Example: For $G = GL_2$, the character $\alpha = (1, -1) \in \mathbb{Z}^2 \cong X^\bullet(T)$ is a root with root subgroup U_α the standard unipotent subgroup. The coroot $\check{\alpha}$ is given by

$$t \mapsto \begin{pmatrix} t & \\ & 1/t \end{pmatrix}.$$

Root datum: See this handout for reference. The definition is given in Wikipedia. Denote $V := \mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$, we call (Φ, V) the associated root system to the root datum. Note that V could be strictly smaller than $X^* \otimes_{\mathbb{Z}} \mathbb{R}$.

Theorem (Chevalley): To a reductive group G the quadraple $R_G := (X^*(T), \Phi, X_*(T), \check{\Phi})$ is a reduced root datum. Moreover, $G_1 \cong G_2$ iff $R_{G_1} \cong R_{G_2}$. Finally, let R be a reduced root datum, then there exists a connected reductive group G such that $R \cong R_G$.

Remark: $Z(T) = T$ for connected reductive G. The Weyl group $W_G \cong N(T)/Z(T) \cong N(T)/T$ is isomorphic to $\langle s_\alpha : \alpha \in \Phi \rangle \subset Aut(X^*(T))$. If we fix a Borel B containing T, then we can define $\Phi^+ := \{\alpha \in \Phi : U_\alpha \subset B\}$ the positive roots. Let $\Delta \subset \Phi^+$ be the set of simple roots. Then the set of reflections s_α for $\alpha \in \Delta$ already generates the Weyl group, and it turns out that it is a Coxeter group.

Example: For $G = SL_2$, then $\Phi = (\mathbb{Z}, \pm 2, \mathbb{Z}, \pm 1)$. For $G = PGL_2$, it is $(\mathbb{Z}, \pm 1, \mathbb{Z}, \pm 2)$ (we have

$$T = \begin{pmatrix} t & \\ & 1 \end{pmatrix}$$

and

$$\alpha : \begin{pmatrix} t & \\ & 1 \end{pmatrix} \mapsto t$$

, and

$$\check{\alpha} : t \mapsto \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} = \begin{pmatrix} t^2 & \\ & 1 \end{pmatrix}$$

For $G = GL_n$, we have $\alpha_{ij} t \mapsto t_i t_j^{-1}$. Note that $V = \mathbb{R}\langle\Phi\rangle$ is a codimension one subspace of $X^*(T) \otimes \mathbb{R}$ (dimension n). For $G = Sp_4$, if

$$t = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1} \end{pmatrix}$$

, then we have

$$\alpha : t \mapsto t_1 t_2^{-1}$$

$$\beta : t \mapsto t_2^2$$

$$\alpha + \beta : t \mapsto t_1 t_2$$

$$2\alpha + \beta : t \mapsto t_1^2$$

with the corresponding root subgroup given previously The root system is type $B_2 = C_2$.

Construction of finite Chevalley group: Recall if the base field is complex, the Lie algebra of G decomposes as $\mathfrak{g} = \mathfrak{t} \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ where $\mathfrak{t}$ is a Cartan subalgebra. For any $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$, let ℓ be the greatest integer such that $\beta - \ell\alpha \in \Phi$. There exists a basis $\{h_1, ...h_r\} \sqcup \{e_\alpha \in \mathfrak{g}_\alpha\}$ satisfying

1. $[h_i, h_j] = 0$;
2. $[h_i, e_\alpha] = \alpha(h_i) e_\alpha$;
3. $[e_\alpha, e_{-\alpha}] = h_\alpha \in \mathbb{Z}\langle h_1, ...h_r\rangle$;
4. $[e_\alpha, e_\beta] = \pm(\ell + 1) e_{\alpha+\beta} \quad \alpha + \beta \in \Phi$;
5. $[e_\alpha, e_\beta] = 0 \quad \alpha + \beta \notin \Phi$.

The upshot is that we can define $\exp(\lambda e_\alpha) = \sum_{n \geq 0} \frac{ad(\lambda e_\alpha)^n}{n!}$ since $ad(\lambda e_\alpha)$ is nilpotent. The integral property of Chevalley basis gurantees that $\exp(\lambda e_\alpha)(e_\beta) = e_\beta \pm \lambda(\ell + 1) e_{\beta+\alpha} \pm \lambda^2 \frac{(\ell+1)(\ell+2)}{2!} e_{\beta+2\alpha} + ..$ where all coefficients $(\ell + 1)$, $\frac{(\ell+1)(\ell+2)}{2!}$... are in $\mathbb{Z}$. Thus $\exp(\lambda e_\alpha) \in Aut(\mathfrak{g}) \cong GL_n(k)$ is

a matrix with entries are polynomials in λ with coefficients in $\mathbb{Z}$. Therefore we can define $G_{ad}(k)$ to be the subgroup generated by $\exp(\lambda e_\alpha)$ for $\lambda \in k$ and $\alpha \in \Phi$.

Chevalley proved that if $k = \overline{k}$ then $G_{ad}(k)$ is an adjoint group of type $\mathfrak{g}$. If $k = \mathbb{F}_q$, then $G_{ad}(\mathbb{F}_q)$ is a finite simple group if $|\mathbb{F}_q| > 5$. The group G_{ad} is defined over $\mathbb{F}_q$ since the Frobenius morphism takes $G_{ad}(k)$ to itself $(\exp(\lambda e_\alpha)^q = \exp(\lambda^q e_\alpha))$.

There is an alternate characterization of length of an element $w \in W$, which is the number of positive roots α such that $w(\alpha)$ is a negative root. Its geometric meaning is that it is the dimension of the open Schubert variety X_w. More precisely we have $X_w \cong \mathbb{A}^{\ell(w)}$.

Example: For $G = SL_3$, we have $X_{123} = pt$ and $\overline{X_{213}} \cong \overline{X_{132}} \cong \mathbb{P}^1$. Moreover, $\overline{X_{231}}$ and $\overline{X_{312}}$ are the total space of the tautological line bundle over $\mathbb{P}^1$ $(\mathcal{O}(-1))$ and its dual (clear from the flag variety description). Finally $\overline{X_{321}} = G/B$. In this case all the closed Schubert varieties are smooth. But for $G = SL_4$, $\overline{X_{4231}}$ and $\overline{X_{3412}}$ are singular. If we fix a complete flag $\{V_i'\}_{i=0}^4$ then $\overline{X_{4231}} = \{V_1 \subset V_2 \subset V_3 : \dim V_2 \cap V_2' \geq 1\}$. There is a map from $\overline{X_{4231}}$ to $Gr(2,4)$ whose image consists of $\{V_2 : \dim V_2 \cap V_2' \geq 1\}$. Kazhdan-Lusztig noticed in 1983 that for $n \leq 3$, the Jordan-Holder series in principal series of $SL_n(\mathbb{C})$ have multiplicity 1. But for $n = 4$ this fails.

Fact: For SL_n, the closed Schubert variety $\overline{X_w}$ is smooth iff w avoid pattern like 4231 and 3412.

Reference:

1. https://virtualmath1.stanford.edu/~conrad/252Page/handouts/alggroups.pdf
2. https://virtualmath1.stanford.edu/~conrad/249BW16Page/handouts/249B_2016.pdf