Local acyclicity and smooth base change

J'ignore • 22 Sep 2025

Slogan: Smooth base change for torsion abelian etale sheaf is what flat base change for quasi-cohrent sheaf.

Idea: Let $X \to D$ is a morphism of a proper complex analytic variety into the disk. In practice, for t small enough, $f^{-1}((0,t])$ is usually a locally trivial fibration (but maybe not over the entire closed interval $[0,t]$, c.f. degeneration of curves in smooth families). Then $X_t := f^{-1}(t) \to f^{-1}((0,t])$ is a homotopy equivalence. If $j : f^{-1}((0,t]) \to f^{-1}([0,t])$ satisfies $j_*\mathbb{Z} = \mathbb{Z}$ and $R^q j_*\mathbb{Z} = 0$ for $q > 0$, then using the Leray spectral sequence for j we can show $H^*(f^{-1}([0,t]),\mathbb{Z}) \xrightarrow{\cong} H^*(f^{-1}((0,t]),\mathbb{Z})$ is an isomorphism. As a result, we can define a cospecialization map

$$cosp^* : H^*(X_t,\mathbb{Z}) \xleftarrow{\cong} H^*(f^{-1}((0,t]),\mathbb{Z}) \xleftarrow{\cong} H^*(f^{-1}([0,t]),\mathbb{Z}) \to H^*(X_0,\mathbb{Z}).$$

To calculate $R^q j_*\mathbb{Z}$ at a point $x \in X_0$ (the only interesting place), we can take a small ball B_ϵ centered at x of radius ϵ; and for $\eta > 0$ small enough, consider the homology cycle $E = X \cap B_\epsilon \cap f^{-1}(\eta t)$. This is the vanishing cycle at x (think of a loop around a smooth hyperboloid that degenerates into a double cone). Then we have

$$(R^q j_*\mathbb{Z})_x \xleftarrow{\cong} H^q(X \cap B_\epsilon \cap f^{-1}((0,\eta t]),\mathbb{Z}) \xrightarrow{\cong} H^q(E,\mathbb{Z}).$$

Thus the cospecialization morphism is defined as long as $H^0(E,\mathbb{Z}) = \mathbb{Z}$ and $H^q(E,\mathbb{Z}) = 0$ for $q > 0$. We say that f is locally acyclic.

If S is a scheme and $\overline{s} \to S$ is a geometric point. Let $\overline{t} \to Spec(\mathcal{O}^{sh}_{S,\overline{s}})$ be a geometric point, we say $\overline{t}$ is the generisation of $\overline{s}$ and $\overline{s}$ is the specialization of $\overline{t}$.

If $f : X \to S$ is a morphism of schemes and $\overline{x} \to X$ is a geometric point of X, then $\overline{s} = f(\overline{x})$ is a geometric point. Consider the base change $F_{\overline{x},\overline{t}} := Spec(\mathcal{O}^{sh}_{X,\overline{x}}) \times_{Spec(\mathcal{O}^{sh}_{S,\overline{s}})} \overline{t}$, we call it the variety of vanishing cycles. We call f is locally acyclic at $\overline{x}$ if for every $\overline{t}$, the reduced cohomology of the constant sheaf $\mathbb{Z}/n$ on $F_{\overline{x},\overline{t}}$ vanishes for every n invertible in $\kappa(\overline{x})$, i.e. $H^0_{et}(F_{\overline{x},\overline{t}},\mathbb{Z}/n) = \mathbb{Z}/n$ and $H^q_{et}(F_{\overline{x},\overline{t}},\mathbb{Z}/n) = 0$ for $q > 0$, and f is locally acyclic if it is locally acyclic at every $\overline{x}$.

Lemma 1 (Stack project 0GJS): Local acyclicity is closed under quasi-finite base change. More generally, it is closed under base change along $S' \to S$ where $S' = \varprojlim S'_\lambda$ is an inverse limit of quasi-finite S-schemes S'_λ with affine transition morphisms $S_\lambda \to S_\mu$.

The idea (and the proof provided by Deligne) is that every vanishing cycle of f' is a vanishing cycle of f.

Lemma 2 (SGA 4.5, V-3, Lemma 1.5, a bit terse; for the full detail, see section 2.9 of Aaron Landesman's note instead) In the following Cartesian diagram, we have $\epsilon'_*\mathbb{Z}/n = f^*\epsilon_*\mathbb{Z}/n$ and the higher pushforward $R^q\epsilon'_*\mathbb{Z}/n = 0$ for $q > 0$.

Again the proof given by Deligne is a bit terse, and you probably don't understand why he considers using the normalization. From my understanding the importance stems from the fact that the etale local ring of a normal scheme is a domain (by applying this and this).

Using Lemma 2, we can define a cospecialization map

$$cosp^* : H^*(X_{\bar{t}}, \mathbb{Z}/n) \to H^*(X_{\bar{s}}, \mathbb{Z}/n)$$

as follows: Consider the Cartesian diagram. Note that f' is locally acyclic by Lemma 1 (note that we need the more general version). We can define $cosp^*$ by

$$H^*(X_{\bar{t}}, \mathbb{Z}/n) \cong H^*(X, \epsilon'_*\mathbb{Z}/n) \to H^*(X_{\bar{s}}, \mathbb{Z}/n).$$

The first arrow (isomorphism) is due to Lemma 2 (and Leray spectral sequence), which tells us that $R^q\epsilon'_*\mathbb{Z}/n = 0$ for $q > 0$. The second one is because the restriction of $\epsilon'_*\mathbb{Z}/n$ to X_s is $\mathbb{Z}/n$ (by local acyclicity and just put $\bar{t} = \bar{s}$.)

Remark: If $\overline{S}$ is the normalization of S in $\kappa(t)$, and $\overline{X} = X \times_S \overline{S}$ then we have $H^*(X_{\bar{t}}, \mathbb{Z}/n) \cong H^*(\overline{X}, \mathbb{Z}/n)$, which shows the utility of normalization. The importance of normality is manifested in Zariski's main theorem, see the excellent explanation of its underlying geometric content here.

Theorem 3 Suppose S is a locally Noetherian scheme, $\bar{s}$ a geometric point of S and $f : X \to S$ a morphism. We suppose

1. The morphism f is locally acyclic.
2. For every geometric point $\bar{t}$ of $Spec(\mathcal{O}^{sh}_{S,\bar{s}})$ and every $q \geq 0$, the cospecialization map $H^q(X_{\bar{t}}, \mathbb{Z}/n) \to H^q(X_{\bar{s}}, \mathbb{Z}/n)$ is bijective.

Then the canonical homomorphism $(R^q f_*\mathbb{Z}/n)_{\bar{s}} \to H^q(X_{\bar{s}}, \mathbb{Z}/n)$ is bijective for every $q \geq 0$.

The proof uses the following reduction: First the question is a local one so we can suppose $S = \mathcal{O}^{sh}_{S,\overline{s}}$. Then it suffices to show that for every sheaf of $\mathbb{Z}/n$-modules $\mathcal{F}$ on S, the homomophism $\varphi^q(\mathcal{F}) : (R^q f_* f^* \mathcal{F})_{\overline{s}} \to H^q(X_{\overline{s}}, f^*\mathcal{F})$ is bijective.

Every sheaf of $\mathbb{Z}/n$-modules is filtered inductive limit of constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules (stack project, 0F0N). Moreover, every constructible sheaf embeds into a sheaf of the form $\prod i_{\lambda *} C_\lambda$ where $i_\lambda : t_\lambda \to S$ is a finite collection of generisations of s (i.e. geometric point of $Spec(\mathcal{O}^{sh}_{S,\overline{s}})$) and C_λ is a finite free $\mathbb{Z}/n$-module on t_λ(Stack project, 09Z6). Note that Lemma 2 and the condition (b) implies that for $\mathcal{F} = i_{\lambda *} C_\lambda$, the homomorphism $\varphi^q(\mathcal{F})$ is bijective (For $q = 0$, we have

$$(R^0 f_* f^* \mathcal{F})_{\overline{s}} = H^0(X, f^* i_{\lambda *} C_\lambda) = H^0(X, i'_{\lambda *} C_\lambda) = H^0(X_{\overline{s}}, C_\lambda) = H^0(X_{\overline{s}}, f^*\mathcal{F}$$

and for $q > 0$ both sides vanish. Note that condition (b) is crucial, e.g. $Spec(\mathbb{A}^1 \setminus \{0\}) \to Spec(\mathbb{A}^1)$ is smooth hence locally acyclic but at $\overline{s} = 0$ LHS is one-dimensional while RHS is zero-dimensional.) The following purely homological algebra lemma finishes the job:

Lemma 4 If $\mathcal{C}$ is an abelian category in which filtered inductive limit exists, $\varphi^\bullet : T^\bullet \to T'^\bullet$ is a map of δ-functors that vanishes in degrees from $\mathcal{C}$ to Ab commuting with filtered colimits. Suppose there exists two subsets $\mathcal{D}$ and $\mathcal{E}$ of objects of $\mathcal{C}$ such that

a. every object of $\mathcal{C}$ is an filtered colimit of objects in $\mathcal{D}$,

b. every object belonging to $\mathcal{D}$ is a subobject of an object belonging to $\mathcal{E}$.

Then TFAE:

i. $\varphi^q(A)$ is bijective for every $q \geq 0$ and for every $A \in Ob(\mathcal{C})$.

ii. $\varphi^q(M)$ is bijective for every $q \geq 0$ and every $M \in \mathcal{E}$.

The proof is by induction on q and a repeated use of five-lemma.

We deduce two corollaries from this theorem:

Corollary 5: If $S = Spec(\mathcal{O}^{sh}_{S,\overline{s}})$, and $f : X \to S$ is a locally acyclic morphism. Suppose for every geometric point $\overline{t}$ of S the fiber $X_{\overline{t}}$ is acyclic (i.e. $\tilde{H}^q(X_{\overline{t}}) = 0$). Then we have $f_*\mathbb{Z}/n = \mathbb{Z}/n$ and $R^q f_q \mathbb{Z}/n = 0$ for $q > 0$.

Corollary 6: Composite of locally acyclic morphisms are locally acyclic. More precisely, if $f : X \to Y$ and $g : Y \to Z$ are morphisms of locally noetherian schemes. If f and g are locally acyclic, so is $g \circ f$.

To see that corollary 6 follows from corollary 5, we can suppose X, Y and Z are strictly local and f and g are local morphisms. We need to show if $\overline{z}$ is a geometric point of Z, then we have $\tilde{H}^q(X_{\overline{z}}, \mathbb{Z}/n) = 0$. Since g is locally acyclic, we have $\tilde{H}^q(Y_{\overline{z}}, \mathbb{Z}/n) = 0$. In addition the morphism $f_{\overline{z}} : X_{\overline{z}} \to Y_{\overline{z}}$ are locally acyclic and the geometric fibers are ayclic because f is locally acyclic. Then by corollary 5, we have $R^q f_{\overline{z}*}\mathbb{Z}/n = 0$ for $q > 0$ and $f_{\overline{z}*}$ are the constant sheaf $\underline{\mathbb{Z}/n}$ on $Y_{\overline{z}}$. We can now conclude using Leray's spectral sequence.

Theorem 7 Smooth morphisms are locally acyclic.

Since the assertion is local for the etale topology on X and S, we can suppose $X = \mathbb{A}^d_S$. By passage to the limit, we can suppose S is Noetherian and transitivity of local acyclicity allows us to further reduce to the case that $d = 1$. Renaming $S = Spec(A)$ and $X = SpecA\{T\}$, where $A\{T\}$ is the henselization of $A[T]$ at $T = 0$, our task is to show the geometric fibers of $X \to S$ are acyclic.

If $\overline{t}$ is a geometric point of S, the fiber $X_{\overline{t}}$ is projective limit of affine smooth curves on $\overline{t}$, and so $H^q(X_{\overline{t}}, \mathbb{Z}/n) = 0$ for $q \geq 2$ by the theory of cohomology of curves. Thus we only need to show $H^0(X_{\overline{t}}, \mathbb{Z}/n) = \mathbb{Z}/n$ and $H^1(X_{\overline{t}}, \mathbb{Z}/n) = 0$.

To show $H^0(X_{\overline{t}}, \mathbb{Z}/n) = \mathbb{Z}/n$, it reduces to the following proposition in commutative algebra:

Proposition 8 If A is a strictly local Henselian ring, $S = Spec(A)$ and $X = SpecA\{T\}$, then the geomtric fibers of $X \to S$ are connected.

By passage to the limit we can reduce to the case that A is a strictly Henselization of a finitely generated $\mathbb{Z}$-algebra. The importance of this reduction is that A will then be an excellent ring. What do we gain? Note that it suffices to show for every $t' = Spec(k')$ where k' is a finite separable subextension of $\kappa(\overline{t})/\kappa(t)$, the fiber $X_{t'}$ is connected. For this we want to reduce to A normal, becuse then $A\{T\}$ will be normal and every localization at a prime will then be a normal domain, so its spectrum $X_{t'}$ is integral, in particular connected. To reduce to this case it essentially boils down to verifying $A\{T\} \otimes_A A' \xrightarrow{\cong} A'\{T\}$ is an isomorphism where A' is the normalization of A in k', and since A is excellent, A' will be finite over A. The claim then follows by observing the ring on the left is Henselian local and filtered colimit of etale local algebras on $A'[T] \cong A[T] \cong_A A'$. (For details see Lemma 2.6.2 of Aaron Landesman's note).

To show $H^1(X_{\overline{t}}, \mathbb{Z}/n) = 0$, it suffices to prove the following which is a restatement using the torsor interpretation of H^1.

Proposition 9 If A is a strictly local Henselian ring, $S = Spec(A)$ and $X = SpecA\{T\}$, and $\bar{t}$ a geometric point of S. Then every $\mathbb{Z}/n$-torsor over $X_{\bar{t}}$ in the etale topology is trivial if $n \neq 0$ in the residue field of A.

The assumption that $n \neq 0$ in the residue field of A is necessary, c.f. the Artin-Schreier cover $Speck\{T\}[x]/(x^p - x - T) \to Spec(k\{T\})$ is a nontrivial connected $\mathbb{Z}/p$-torsor when k is a separably closed field of characteristic p.

http://math.stanford.edu/~conrad/papers/nagatafinal.pdf

Pushforward along finite morphism is exact: https://stacks.math.columbia.edu/tag/03QP

Relative normalization: https://stacks.math.columbia.edu/tag/0BAK

Sorry for very quickly glossing over the last part of smooth base change, which is proving smooth maps are locally acyclic. Let me try to explain the $q = 1$ case better.

In the previous steps, we have reduced to the case that $S = \mathrm{Spec}(A)$ for A strictly Henselian (and excellent, which is the case if A is the strict local ring of a finitely generated $\mathbb{Z}$-algebra) and $X = Spec(A\{T\})$, where the notation $A\{T\}$ means we are taking the strict Henselization of $A[T]_{(0)}$. To show $H^1_{\text{ét}}(F_{\overline{x},\overline{t}}, \mathbb{Z}/n\mathbb{Z})$ vanishes, it amounts to the following statement (for $\overline{x}$ lying above 0 WLOG):

> Let $\bar{t}$ be a geometric point of S and $X_{\bar{t}}$ be the corresponding geometric fiber. Then every connected étale torsor $\tilde{X}_{\bar{t}}$ over $X_{\bar{t}}$ of order n coprime to the residual characteristic of A is trivial.

1. The first reduction we can make is to reduce to the case that the image t of $\bar{t}$ in S is the generic point by replacing S by $S' = \mathrm{Spec}(A')$ is the closure of t in S.
2. We can replace A by the normalization of A in a finite separable extension $\kappa(t')$ of $\kappa(t)$ in $\kappa(\bar{t})$ (thus we can S normal) because:

i. I can find $\kappa(t')$ and étale torsor $\tilde{X}'$ over $X' := X \times_t t'$ such that $\tilde{X}_{\bar{t}} = \tilde{X}' \times_{t'} \bar{t}$. The connnectedness of $\tilde{X}'$ follows from the connectedness of $\tilde{X}_{\bar{t}}$.

ii. I can take the normalization B of A inside $\kappa(t')$ (which will be finite over A if A is excellent) and B will also be a strict Henselian local ring. I also replace X by $X \times_S \mathrm{Spec}(B)$ (which doesn't change X' and $\tilde{X}'$).

In the following we rename t' as t, so X', $\tilde{X}'$ become X_t and $\tilde{X}_t$.

3. By spreading-out we can find a dense open $U \subset S$ such that $\tilde{X}_t$ extends to an étale torsor $\tilde{X}_U$ over $X_U = X \times_S U$.

4. The first key point is that upon further base changing S to the normalization in a finite separable extension (which is harmless by point 2), I can arrange $S \setminus U$ has codimension at least 2 in S:

 i. Let s be a codimension-1 point in $S \setminus U$, Let $V = \mathrm{Spec}(\mathcal{O}_{S,s})$ (note that $\mathcal{O}_{S,s}$ is a DVR since S is normal). By abuse of notation we still call t the generic point of V. Our goal is to extend the étale torsor $\tilde{X}_t$ of X_t across X_V after some base change.

 ii. Applying Abhyankar's lemma to V, we see after base changing from V to $V_1 := \mathrm{Spec}(\mathcal{O}_{S,s}[\pi^{1/n}])$ where π is a uniformizer for $\mathcal{O}_{S,s}$, we can extend $\tilde{X}_1 := \tilde{X}_t \times_{X_t} X_{t_1}$ (where t_1 is the generic point of V_1) across the generic point of X_{s_1}, where s_1 denotes the unique closed point of V_1.

 iii. By Zariski-Nagata in dimension 2 (which says that ramification is a codimension-1 phenomenon) applied to the 2-dimensional regular local ring X_{V_1}, we see that the normalization of X_{V_1} in $\tilde{X}_1$ is a finite étale cover over X_{V_1} (because the only codimension-1 point of X_{V_1} for which étaleness is not automatic is the generic point of X_{s_1} for which we have taken care of in 4(ii)).

5. By invoking step 2 again (enlarging $\kappa(t)$ if necessary) we can assume $\tilde{X}_U$ is trivial above the subscheme $(T = 0) \cap X_U = U$.

6. To finish the proof that $\tilde{X}_U$ is trivial, let $R = \Gamma(\tilde{X}_U, \mathcal{O})$ and it suffices to show R is finite étale over $A\{T\}$ (since $A\{T\}$ is strictly Henselian, we must have $R = A\{T\}$). Note that we can check it by base changing everything from X to $\widehat{X} := \mathrm{Spec}(A[[T]])$, since $\widehat{X}$ is faithfullly flat over X. Thus our task now is to show $\widehat{R} := R \otimes_{A\{T\}} A[[T]]$ is finite étale over $A[[T]]$. If we let V_m and $\tilde{V}_m$ be the subscheme of $\widehat{X_U}$ and $\widehat{\tilde{X}_U}$ defined by $T^{m+1} = 0$, then by hypothesis $\tilde{V}_0$ is trivial over V_0, i.e. $\tilde{V}_0 \cong V_0^n$. The key point is that the same holds for $\tilde{V}_m$ over V_m, since étale sites are insensitive to nilpotents. Let φ be the following composite as a map of $A\{T\}$-algebra:

$$\Gamma(\widehat{\tilde{X}_U}, \mathcal{O}) \to \varprojlim_m \Gamma(\tilde{V}_m, \mathcal{O}) \cong (\varprojlim_m \Gamma(V_m, \mathcal{O}))^n$$

Since the complement of U has codimension ≥ 2, we can apply Hartogs's theorem and get $\Gamma(V_m, \mathcal{O}) = A[T]/(T^{m+1})$, so the target of φ can be identified with $A[[T]]^n$. The domain of φ is $\widehat{R}$ by flat base change. Hence it suffices to show φ is an isomorphism, which follows from checking it over U.