

Adic spaces: Chapter 2 and 3 of Scholze-Weinstein's Berkley

Lecture's on p -adic geometry

J'ignore • 21 Sep 2025

Recall the definition of a formal scheme $\mathrm{Spf} A$ of an adic ring A (topological ring carrying the I -adic topology for some ideal I , called an ideal of definition, which is far from unique): The underlying set is the set of all open prime ideals (This agrees with $\mathrm{Spec} A/I$). For any $f \in A$, the nonvanishing locus $D(f)$ generate the topology of $\mathrm{Spf} A$. The structure sheaf is given by the completion of the structure sheaf of $\mathrm{Spec} A$, i.e. $\mathcal{O}_{\mathrm{Spf} A}(D(f)) = \widehat{A[1/f]}$, the I -adic completion of $A[1/f]$ (Note that formal schemes are not schemes). Intuitively, it consists of all infinitesimal thickenings of the usual spectrum $\mathrm{Spec}(A/I)$. A formal scheme is a ringed space locally of the form $\mathrm{Spf} A$ for an adic ring A . (Scholze said that we take I to be finitely generated so that $\widehat{M}/I\widehat{M} \cong M/IM$ holds (stack project, tag 05GG), which implies $\widehat{M}$ is I -complete. The key is that to show $I\widehat{M} = \widehat{IM}$, and eventually boils down to the fact that finite direct sum commute with inverse limit)

There is another category of rigid analytic spaces. This consists of spaces locally isomorphic to K -affinoid spaces in some Grothendieck topology. As it appears, formal schemes are more algebro-geometric objects whereas rigid-analytic spaces are more analytic. There is a 'generic fiber functor' from a certain class of formal schemes over $\mathrm{Spf} \mathbb{Z}_p$ to rigid analytic spaces over $\mathbb{Q}_p$ due to Berthelot (see [here](#))

The goal is to define a larger category of objects that contain both of them. Just as formal schemes are built out of affine formal schemes associated to adic rings, and rigid-analytic spaces are built out of affinoid spaces associated to affinoid algebras, adic spaces are built out of affinoid adic spaces, which are associated to pairs of topological rings (A, A^+) (where A^+ plays a secondary role). The affinoid adic space associated to such a pair is written $\mathrm{Spa}(A, A^+)$, the adic spectrum.

A ring is Huber if A contains an open subring A_0 that is adic with respect to a finitely generated ideal of definition I . We called A_0 a ring of definition. Note that this is a property of A , we may ask which class of subrings of A can serve

as a ring of definition? One characterization is open and boundedness (i.e. S is bounded if for any open neighborhood U of 0, there exists open neighborhood V of 0 such that $VS \subset U$.)

There are an important classes of Huber rings. A is Tate if it contains a topological nilpotent unit (such unit is called pseudo-uniformizer). The name is reminiscent of Tate algebra and indeed Tate rings arise from inverting a nonzerodivisor (more precisely see Proposition 2.2.6.), so examples are $\mathbb{Q}_p$ and $\mathbb{Q}_p\langle T \rangle$ (the latter has ring of definition $\mathbb{Z}_p\langle T \rangle$), but $\mathbb{Z}_p$ and $\mathbb{Z}_p[[T]]$ are not Tate.

If A is a complete (complete means complete and separated, i.e. the I -adic topology is Hausdorff and complete) Tate ring and g a pseudo-uniformizer, we can define a norm on A (the escape norm). Under this norm, A is a Banach ring whose unit ball is A^0 . This construction gives an equivalence of categories between the category of complete Tate rings (with continuous homomorphisms), and the category of Banach rings A that admit an element $g \in A$, $|g| < 1$ such that $|g||g^{-1}| = 1$ (with bounded homomorphisms).

A slight generalization of the Tate condition has recently been proposed by Kedlaya. A Huber ring A is analytic if the ideal generated by topologically nilpotent elements is the unit ideal.

As we remark the ring of definition (open and bounded subrings) is not unique, but there is a canonical subring of a Huber ring, that is, the subring A° of power-bounded elements (easy to check that it is a subring). For example, if $A = \mathbb{Q}_p\langle T \rangle$, then $A^\circ = \mathbb{Z}_p\langle T \rangle$ is a ring of definition. However, if A is nilpotent, e.g. $A = \mathbb{Q}_p\langle T \rangle/T^2$, then $A^\circ = \mathbb{Z}_p \oplus \mathbb{Q}_p\langle T \rangle$, which cannot be a ring of definition (since it is not even bounded). We have the following propositions:

1. Any ring of definition A^0 is contained in A .
2. The ring A is the filtered union of the rings of definition $A^0 \subset A$. (The word filtered here means that any two subrings of definition are contained in a third.)

We have another important property of Huber rings: A is uniform if A° is bounded, i.e. a ring of definition.

We remark that if A is separated, Tate, and uniform, then A is reduced.

A subring A^+ is a ring of integral elements if it is open and integrally closed in A , and $A^+ \subset A^\circ$. Note that A^+ contains the set of topologically nilpotent elements by integral closedness, and we can take $A^+ = A^\circ$ (It is easy to check that A° is integrally closed in A).

Now we define the spaces associated to a Huber pair (A, A^+) . It is the set of (equivalence classes of) continuous valuations x such that $|f(x)| \leq 1$ for all $f \in A^+$ (we write $f(x)$ to mean the valuation of f w.r.t. x) with open sets given by $\{x : |f(x)| \leq |g(x)| \neq 0\}$ for $f, g \in A$.

This definition combines features from both algebraic geometry and nonarchimedean analysis. Huber prove that $\mathrm{Spa}(A, A^+)$ is spectral, i.e. homeomorphic to $\mathrm{Spec}(R)$ for some ring R . See Example 2.3.6 for how $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$ looks like (it includes both points from $\mathrm{Spec}(\mathbb{Z})$ and the p -adic valuations). In general, there is a map $\mathrm{Spec}(R) \rightarrow \mathrm{Spa}(R, R)$ and a map $\mathrm{Spa}(R, R) \rightarrow \mathrm{Spec}(R)$ (sending a valuation to its kernel). The composition is the identity on $\mathrm{Spec}(R)$ and both maps are continuous.

Example 2.3.7 describes the adic space of the Tate algebra $A = K\langle T \rangle$ with $A^+ = \mathcal{O}_K\langle T \rangle$. For details see [Wedhorn](#), example 7.57 or [Morel](#) III.5.2.

Example 2.3.8 explains that for K a discrete field, and A a valuation ring of K . The adic space $\mathrm{Spa}(K, A)$ is nothing but the Zariski-Riemann space, consisting of the set of all valuation rings of K containing A (using the correspondence between valuation rings and valuations).

Proposition 2.3.10 is an important one. It tells us when an element $x \in A$ is actually in A^+ and when x is invertible by looking at $\mathrm{Spa}(A, A^+)$.

Moving on to chapter 3, we define localization of adic spaces and upgrade $X = \mathrm{Spa}(A, A^+)$ from topological spaces to locally ringed spaces. For $T \subset A$ finite subset such that $TA \subset A$ is open, we define the subset

$$U(T/s) := \{x \in X : |t(x)| \leq |s(x)| \neq 0, \forall x \in T\}$$

Such subsets are called rational open subsets (they are open because they are intersection of finitely many open subsets). We can check that intersection of two rational open is rational. The following is an important theorem that allows us to define structure sheaf of an adic space:

Let $U \subset \mathrm{Spa}(A, A^+)$ be a rational subset. Then there exists a complete Huber pair $(A, A^+) \rightarrow (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ such that the map $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \rightarrow \mathrm{Spa}(A, A^+)$ factors over U , and is universal for such maps. Moreover this map is a homeomorphism onto U . In particular, U is quasi-compact.

The proof of this uses Proposition 2.3.10. The idea is that we should take $\mathcal{O}_X(U) := A[1/s]$ and $\mathcal{O}_X(U) :=$ integral closure of $A^+[T/s]$ in $A[1/s]$ (This is one of the reason we want to allow A^+ other than A°). One key thing is that at some point we need to show that multiplication by $1/s$ is continuous with respect to the topology on $A[1/s]$ given by a ring of definition $A_0[T/s]$ with respect to the ideal $IA_0[T/s]$. It boils down to showing TA_0 is open (assuming that TA is open) and this seems to require that I is finitely generated as well.

Reference:

See [this post](#) for a discussion whether rigid analytic spaces are obsolete after adic spaces.

See [this post](#) and [this post](#) for a discussion of the three kinds of non-Archimedean geometry and their relationships.