Lurie's higher topos takeaway

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Below is some highlight from Lurie's higher topos chapter 1:

1. How to avoid circularity in higher category theory: define (∞, n) category as infinity categories in which all k-morphisms are invertible for $k > n$; $(\infty, 0)$ category is ∞-groupoid. We can view (∞, n) category as a category enriched over the category of all $(\infty, n-1)$ category (viewed as an ∞-category by discarding all noninvertible k-morphisms for $2 \leq k \leq n$). This also suggests $(\infty, 1)$ category is category enriched over spaces (if $(\infty, 0)$ category aka ∞-groupoids are spaces), and we can require associativity to be strict (this doesn't make a difference).

2. The main difficulty of working with topological cateogory it is that most natrual construction give rise to $(\infty, 1)$ categories that are only associative up to coherent homotopy, so it is necessary to straighten it to get a strictly associative composition law.

3. The class of weak Kan complexes incorporate both ∞-groupoids and ordinary categories. First, for any space X, the singular complex functor $X \mapsto SingX$ has a left adjoint, namely geometric realization $|K|$ of a singular set K. Moreover, the counit $|SingX| \to X$ is a weak homotopy equivalence. Thus if one is interesting in spaces up to weak homotopy equivalence, one can work with simplicial sets.

4. The singular complex of any space is a Kan complex. Moreover, there is a simple combinatorial recipe to extract homotopy groups from a Kan complex K (turns out to be isomorphic to the homotopy groups of the topological space $|K|$). According to a theorem of Quillen, the singular complex and geometric realization provide mutually inverse equivalences between the homotopy category of CW complexes and the homotopy category of Kan complexes.

5. On the other hand, by taking nerve of a cateogry we also get a simplicial set that satisfies a slightly different condition than Kan complexes. Moreprecisely, the Horn filling condition is now required only for $0 < i < n$ but we in addition require uniqueness of the extension to Δ^n. This in fact characterize simplicial sets that arise from the nerve construction, see Prop. 1.1.2.2.

6. The philosophy of higher category is to think of composition of morphisms not as a function, but as a relation. As such we define an ∞-category as a simplicial set K which satisfies the Horn filling condition for $0 < i < n$, and it is also referred to as weak Kan complexes.

7. When are two topolgical categories equivalent, or when can we call $F : \mathcal{C} \to \mathcal{D}$ an equivalence of categories? We can of course require $Map(X, Y) \to Map(FX, FY)$ to be a homeomorphism. But this is too strong since $Map(X, Y)$ is only defined up to homotopy equivalence. Define the homotopy category (as an ordinary category) $h\mathcal{C}$ as follows: The objects are the same as $\mathcal{C}$, but morphisms are $Mor(X, Y) := \pi_0(Map(X, Y)) = [X, Y]$. A weak homotopy equivalence is one that induces isomorphism on all homotopy groups. CW approximation theorem tells us that any space is weakly homotopy equivalent to a CW complex (and it is unique up to canonical homotopy equivalence), and whitehead theorem says that if a map between two CW complexes is a weak homotopy equivalence, it is a (strong) homotopy equivalence (note that this is not the same as saying two CW complexes with the same homotopy types are homotopy equivalent! For counterexamples see here and here). Thus $\mathcal{H} := hTop$ can be seen as the category obtained from Top by formally inverting all weak homotopy equivalence.

8. The construction can be improved by incorporating higher homotopies groups of $Map(X, Y)$. More precisely, for a space X, let $[X] := X'$ be a CW complex weakly homotopy equivalent to it, then $X \mapsto [X]$ defines a functor from Top to $\mathcal{H}$. Redefine $h\mathcal{C}$ to have the same objects of $\mathcal{C}$ but now $Mor_{h\mathcal{C}}(X, Y) := [Map_{\mathcal{C}}(X, Y)]$. This compatible with the previous definition since $\pi_0 X \simeq Map_{\mathcal{H}}(*, [X])$. Hence now $h\mathcal{C}$ is a category enriched over $\mathcal{H}$. We should think of it as the object which is obtained by forgetting the topological morphism spaces of $\mathcal{C}$ and only remember their (weak) homotopy types. Now define $F : \mathcal{C} \to \mathcal{D}$ to be a (weak) equivalence, if $h\mathcal{C} \to h\mathcal{D}$ is an equivalence of $\mathcal{H}$-enriched categories.

9. To bridge between the two theories of ∞-category, one in terms of topological categories, and the other in terms of weak Kan complexes, we introduce simplicial categories. They are categories enriched over simplcial sets. The aforementioned Quillen equivalence between simplicial sets and topological spaces are proved in Theorem 11.4 of the book Simplicial homotopy theory.

10. To relate simplicial categories to simplical sets, we use the simplicial nerve functor. The nerve of an ordinary category is $Hom(\Delta^n, N(\mathcal{C})) = Hom_{Cat}([n], \mathcal{C})$. However this makes no use of teh

simplical structure of $\mathcal{C}$. The idea to replace the linear ordered set $[n]$ by a thickening $\mathfrak{C}[\Delta^n]$, see Definition 1.1.5.1, the topological space $Map(i,j)$ is homeomorphic to a cube. Moreover, $\mathfrak{C}[J]$ is functorial in J, so it determines a functor from the Δ (viewed as a category) to Cat_Δ by sending Δ^n to $\mathfrak{C}[\Delta^n]$. Now for a simplicial category $\mathcal{C}$, define $N(\mathcal{C})$ as $Hom(\Delta^n, N(\mathcal{C})) = Hom_{Cat_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C})$.

11. The functor $\mathfrak{C}$ extends uniquely to a colimit preserving functor $Set_\Delta \to Cat_\Delta$ from the category of simplicial sets to that of simplicial categories by formal non-sense. By construction it is left adjoint to the simplicial nerve functor N. See Example 1.1.5.9 for an explicit description of $\mathfrak{C}[NA]$ for a post A (so the nerve NA is a simplicial set)

12. The upshot is that if $\mathcal{C}$ is a simplicial category having the property that $Map_{\mathcal{C}}(X,Y)$ is a Kan commplex for every $X, Y \in \mathcal{C}$. Then the simplical nerve $N(\mathcal{C})$ is an ∞-category (weak Kan complex). One corollary is that the topological nerve of a topological cateogory is an ∞-category (since singular sets are Kan complexes).

13. An important theorem is Theorem 1.1.5.13, which says that if $\mathcal{C}$ is a topological category, then taking the topological nerve $N(\mathcal{C})$ (which yields a simplicial set) then applying the $\mathfrak{C}$ functor (which then yields a simplical category) and finally taking the geometric realization of the morphism set, is weakly homotopy equivalent to the morphism space of $\mathcal{C}$ via the counit map. This theorem underlies the equivalence of homotopy cateogories among the three models (topological categories, simplicial categories and simplicial sets) of ∞-cateogies.

14. Generalizing notions from classical category theory to higher categories: The opposite of an ∞-categories is simply reversing the order of the face and degeneracy maps.

15. The unstraightening functor is an ∞-cateogory analogue of Grothendieck construction. It is easier to describe the ∞-version of its left adjoint (the straightening functor), and then use the adjoint functor theorem to construct it. In original Grothendeick construction $\int : Func(\mathcal{C}^{op}, Cat) \to Cat/\mathcal{C}$. The essential image of this functor consists of Grothendieck fibrations and this establishes an equivalence of 2-categories $Func(\mathcal{C}^{op}, Cat) \xrightarrow{\cong} Fib(\mathcal{C})$. When restricted LHS to takes value in *Grpd*, it identifies with the Grothendieck fibrations in groupoids. What should the left adjoint F be? Let $p : E \to \mathcal{C}$ be a functor. Then Fp is a functor from $\mathcal{C}$ to Cat taking c to the the comma category $(e, c \to p(e))$. To generalize this to the ∞-cateogry setting, we can reprhase it in terms of the cone construction. See here for more details. One important application of Grothendieck construction is to show that

every presheaf is a colimit of representables, i.e. the density theorem. See also this post for an explicit description of unstraightening. For $n = 1$ we get the comma category construction.

16. For $\phi = id$ and $S = \{x\}$, we can view St_ϕ as a functor from simplicial sets to simplicial sets. This can be identified with some geometric realization functor $||_{Q^\bullet}$ for some cosimplicial object $Q^\bullet$ (a cosimplicial object $C^\bullet$ in some category $\mathcal{C}$ is a functor from $\Delta \to \mathcal{C}$ where Δ is the category of $\{\Delta^n\}$, which extends to a colimit-preserving functor $F : Set_\Delta \to \mathcal{C}$). The cosimplicial object $Q^\bullet$ can be explicitly defined by first defining a
$$J^\bullet = (\Delta^\bullet \star \{y\}) \bigsqcup_{\Delta^\bullet} \{x\}$$
and then defining $Q^\bullet = Map_{\mathfrak{C}[J^\bullet]}(x, y)$. The motivation for $Q^\bullet$ is that if $\mathcal{C}$ is a simplicial category and $S = N(\mathcal{C})$ its simplicial nerve, then for every pair of vertices $x, y \in S$, the n-simplices of the right mapping space $Hom^R_S(x, y)$ are the same as a map $\mathfrak{C}[J^n]$ into $\mathcal{C}$ carrying x to x and y to y. This is simply the data of a map of simplicial sets $Q^n \to Map_{\mathcal{C}}(x, y)$.

See the comment to this question for why s/u are important. Quote: The problem is that mapping spaces are not automatically functorial (essentially because you don't have a strict composition in quasicategories), so the usual formula for the represented presheaf doesn't define a functor. The construction of a "mapping space" functor and everything obtained from it, like the Yoneda embedding, is one of the main applications of straightening.

(to be continued)