Pro-unipotent completion

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The motivation comes from rational homotopy theory. The attaching map of $S^{n+m-1} \to S^n \wedge S^m$ (from the cell structure of $S^n \times S^m$) gives a graded lie algebra structure on $\pi_{*-1}(X) \otimes \mathbb{Q}$: If $\alpha \in \pi_m, \beta \in \pi_n$, $[\alpha, \beta] \in \pi_{m+n-1}$ is $S^{n+m-1} \to S^n \wedge S^m \to X$ (note that in the case $n = m = 1$ we get $aba^{-1}b^{-1}$). If X is simply connected, then $H_*(\Omega X, \mathbb{Q}) \cong \mathcal{U}(\pi_*(\Omega X) \otimes \mathbb{Q})$ as a Hopf algebra.

One of the tool for studying rational homotopy group is the homology of the loop space is the Eilenberg-Moore spectral sequence (fiber product analogue of Kunneth formula). As a special case, we have $H_*(\Omega X) \Leftarrow \mathrm{Cotor}^{H_*(X)}_{*,*}(k, k)$, converge strongly if X is simply connected (more generally, it states $H^*(E_f) \Leftarrow Tor_{H^*(B)}(H^*(E), H^*(X)) = H^*(E) \otimes^L H^*(X)$, take $X = *, E = PB$, then get the above special case.

This goes wildly wrong if $X = K(G, 1)$ and G is finite (LHS is just $\mathbb{Q}$ at degree 0 since $\mathrm{Ext}_{H^*(BG)}(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q}$, but $H_*(\Omega BG, \mathbb{Q}) \cong H_*(G, \mathbb{Q}) = \mathbb{Q}[G]$, so simply connectedness is crucial).

The claim is that the best approximation to $\pi_1(X, *)$ that $H_*(X, *)$ can see is the pro-unipotent (malcev) completion. What is it? It is the initial object in the category of pro-unipotent groups (inverse limit of unipotent groups) receiving a group homomorphism from G (this depends on field k).

Here is Quillen's Construction of pro-unipotent completion of a group: Let $I = ker(\epsilon)$ augmentation ideal of the group ring $k[G]$ (generated by $g - 1$ for $g \in G$). The completed group ring $\widehat{k[G]} := \varprojlim_n k[G]/I^n$ is a completed Hopf algebra (using the completed tensor product). Define $\widehat{G}$ to be the group-like elements x and such that $\epsilon(x) = 1$.

> If I/I^2 is a finite dimensional vector space over k (e.g. G is finite generated), then $\widehat{G}$ is pro-unipotent completion.

For a pro-unipotent group $\varprojlim G_\alpha$ its Lie algebra is easy to compute: it is just the inverse limit of the individual Lie algebra, and we have mutually inverse map $\exp$ and $\log$ (because of the unipotent assumption). Note that we can also make sense of $g^\lambda = \exp(\lambda \log(g))$ for $\lambda \in k$ by using the formula $(1-(1-g))^\lambda$ and the binomial formula (which makes sense since $1-g$ is nilpotent).

Assume $\mathrm{char}\ k = 0$. We give some example:

1. $G = \mathbb{Z}$, $k[\mathbb{Z}] \cong k[t, t^{-1}]$, and $I = (t-1)$. Setting $T = t-1$, we get $\widehat{k[\mathbb{Z}]} \cong \varprojlim k[T]/(T^n) = k[[T]]$. $\Delta(T) = T \otimes T + 1 \otimes T + T \otimes 1$; t^λ is also group-like. The pro-unipotent group is isomorphic to k.

2. $G = \mathbb{Z}/m$, I=$(t-1)$, since $(t-1)^2 = t^m + t^2 - 2t = t(t^{m-1} + t - 2) = t(t-1)(t^{m-2} + ... + t + 2)$, so $I^2 = I$, so $\widehat{k[\mathbb{Z}/m]} = k[\mathbb{Z}/m]/I = k$ so the pro-unipotent completion is trivial.

Thus the intuition is that pro-unipotent completion can only see non-torsion phenomenon. The lie algebra of the pro-unipotent completion is the primitive elements of the completed group algebra and we have $G = \{\exp(x) : x \in \mathfrak{g}\}$.

3. $G = F_n$ free group on n generators. The group ring isomorphic to $k\langle t_i, t_i^{-1}\rangle$, $I = (t_i - 1)$, $k[F_n]/I^m = T(k\{T_i\})/deg_{\geq m}$. so $\widehat{F_n} = \exp(\widehat{lie_n})$ completed free lie algebra. Generally if $G = \langle t_i | R_j(t_i)\rangle$ and $\widehat{G} = \exp(\widehat{\mathfrak{g}})$ and $\widehat{g} = \widehat{lie_n}/(\log R_j(t_i))$.

The difference between profinite completion (which is inverse limit over G/N taken over all finite index subgroup N) and pro-unipotent completion is that we are completing along $g^n = 1$ versus $(g-1)^n = 0$.