

Automorphic lifting lecture 4-6

J'ignore • 11 Sep 2025

Some notation: $\mathbb{A}^\infty = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$ (upper script means away from a certain place); $\mathbb{A} = \mathbb{R} \times \mathbb{A}^\infty \cong \prod' \mathbb{Q}_v$; For general number field F , we define $\mathbb{A}_F = \mathbb{A} \otimes_{\mathbb{Q}} F \cong \prod' F_v$, and $G = \text{Res}_{F/\mathbb{Q}} GL_n$, the Weil restriction of GL_n from F to \mathbb{Q} (representable by choosing generators and relation of F over \mathbb{Q} , but it is not true in general, see [this post](#); The original reference is Grothendieck's Bourbaki lecture no. 221 (May 1961), Techniques ... IV: les schemas de Hilbert, §4 c. Note that Grothendieck denotes the Weil restriction by). We equip $GL_n(\mathbb{A}_F)$ by viewing it as a subset of $\mathbb{A}_F^{n^2+1}$ (last coordinate being $1/\det$). For why we topologize it this way see [this](#).

Exercise: Check the two topologies on ideles agree. (r is close to r' and r^{-1} is close to r'^{-1} , the problem are there are non-units that converge to 1)

Define the ideles class group $\mathcal{C}_F = F^\times \backslash \mathbb{A}_F^\times$; the group of fractional ideals $\mathcal{J}_F := \mathbb{A}_F^{\infty, \times} / \prod_{v \nmid \infty} \mathcal{O}_{F_v}^\times \cong \bigoplus_{v \nmid \infty} (F_v^\times / \mathcal{O}_{F_v}^\times)$. As an abstract group this is $\bigoplus \mathbb{Z}$, infinite direct sum of \mathbb{Z} with indexing set equal to all finite places. The ideal class group $Cl_F = F^\times \backslash \mathcal{J}_F = F^\times \backslash \mathbb{A}_F^{\infty, \times} / \prod_{v \nmid \infty} \mathcal{O}_{F_v}^\times$. The relationships are shown in this [diagram](#).

For each modulus \mathfrak{m} we get a ray class group $Cl_F^\mathfrak{m}$ similar to the class group Cl_F . To define this first let $\mathcal{U}_\mathfrak{m} = \prod_{v \nmid \infty, r_v > 0} (1 + \mathfrak{p}_v^{r_v} \mathcal{O}_{F_v}) \times \prod_{v \nmid \infty} \mathcal{O}_{F_v}^\times$. Such $\mathcal{U}_\mathfrak{m}$ are cofinal among all open compact subgroups. At infinity places consider $G(\mathbb{R})^\circ$, which is product of copies of $\mathbb{R}_{>0}$ and \mathbb{C}^\times . Finally let $Cl_F^\mathfrak{m} = G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\mathbb{R})^\circ \mathcal{U}_\mathfrak{m}$.

Close finite index subgroups (same as open compact if we restrict to norm-one element of the adeles) of \mathcal{C}_F are exactly those containing for some \mathfrak{m} as above. More generally for each open compact $U \subseteq G(\widehat{\mathbb{Z}})$, it is still true that $G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\mathbb{R})^\circ U$ is a finite set (weak approximation), and it is a group when G is commutative. If K/F is a finite Galois extension, then we get a norm map $N_{K/F} : \mathbb{A}_K^\times \rightarrow \mathbb{A}_F^\times$, which induces a map $\mathcal{C}_K \rightarrow \mathcal{C}_F$. Note that the image is an open subgroup of finite index.

Statement of Artin reciprocity:

- There exists canonical global Artin reciprocity map or norm residue symbol: $(\cdot, K/F) : G(\mathbb{A}) = \mathcal{C}_F = F^\times \backslash \mathbb{A}_F^\times \rightarrow \text{Gal}(K/F)^{ab}$ (topological abelianization if K/F is not finite, which is the maximal abelian Hausdorff quotient) and it is surjective (specific to number fields) with kernel or $N_{K/F}\mathcal{C}_K$.
- Existence of class fields: The assignment $K \mapsto N_K = N_{K/F}(C_K)$ defines a bijection from abelian extension K/F to closed subgroup of finite index N of \mathcal{C}_F . For example $N = \mathcal{C}_F^{\mathfrak{m}}$ corresponds to the ray class field mod \mathfrak{m} .
- Local-global compatibility: For each place v of F with algebraic closure \overline{F}_v of F_v extending \overline{F} , there is also a canonical local Artin reciprocity map $\text{Art}_{F_v} : F_v^\times \rightarrow \text{Gal}(\overline{F}_v/F_v)^{ab}$ with same existence of finite abelian extension of F_v for each open subgroup of finite index such that the **diagram** commutes.

A more natural formulation is to use the Weil group. This is done in Tate's Corvallis paper '[On Number-theoretic background](#)'. Some remark:

- The weil group W_F satisfies some natural axioms. For example, we should have an isomorphism (as topological group) $W_F^{ab} \cong \mathcal{C}_F$ if F is global and F^\times if F is local. We also should be able to write W_F as a projective limit $W_F \cong \varprojlim W_{K/F}$ where $W_{K/F} := W_F/W_K^c$. If F is local, W_F is coming from the **diagram**. To introduce a filtration on W_F , we consider $G_F \rightarrow \text{Gal}(K^{ab}/F) \rightarrow \widehat{\mathbb{Z}}$. The inverse image of \mathbb{Z} in G_F is W_F , and define $W_{K/F} :=$ inverse image of \mathbb{Z} in $\text{Gal}(K^{ab}/F)$. Note that $\text{Gal}(K^{ab}/F)$ surjects onto $\text{Gal}(K/F)$ with kernel $\text{Gal}(K^{ab}/K)$. Thus $W_{K/F}$ is an extension of $\text{Gal}(K/F) \cong W_F/W_K$ by W_K^{ab} , and from this it is not hard to check that it is isomorphic to W_F/W_K^c (This viewpoint also suggests that we can use galois cohomology to construct $W_{K/F}$ and therefore W_F).
- If K is local and $a \in K^\times$, then $\text{Art}_v^t(a)|_{\overline{k}} = x \mapsto x^{|a|_K}$, so uniformizers correspond to inverse of the usual Frobenius automorphism $x \mapsto x^q$.

For a clean explanation of how Artin reciprocity implies quadratic reciprocity, see [this wikipedia page](#). Essentially it has to do with how primes split in quadratic fields and cyclotomic fields and the compatibility of the Artin symbol.

Some Baire category arguments: Any continuous representation $\rho : \Gamma \rightarrow \text{GL}_n(\overline{\mathbb{Q}_p})$ where Γ is a compact topological group, factors through some $\Gamma \rightarrow \text{GL}_n(E_v)$ where E_v is a finite extension of \mathbb{Q}_p . The proof is to assume the following fact, the number of finite extensions E of \mathbb{Q}_p is countable (since a finite extension E of \mathbb{Q}_p is completion of a finite extension F over \mathbb{Q} ,

which is **Krasner's lemma** (algebraic extensions of nearby polynomials are equal)). For any such E , $GL_n(E) \subset GL_n(\overline{\mathbb{Q}_p})$ is closed, which implies $\Gamma = \cup_E \Gamma_E$ where $\Gamma_E = \rho^{-1}(GL_n(E))$, there exists E_1 such that Γ_{E_1} has positive Haar measure. Thus $\Gamma = \sqcup g\Gamma_{E_1}$ for finitely many disjoint union. Take E to be finite extension of E_1 given by entries of $\rho(g)$. In particular we can take Γ to be the absolute Galois group.

But why do we consider p -adic representation? Because they appear naturally. An Artin representation is a complex representation of the absolute Galois representation G_F that has finite image, i.e. factors through $Gal(K/F)$ for some finite extension K/F . In this case, once we fix an isomorphism between $\overline{\mathbb{Q}_p} \xrightarrow{\cong} \mathbb{C}$, we get a p -adic representation. The reason why this could be better is that we could put Artin representations into (p -adic) families, whereas it is much harder to take a complex families of Artin representations. The idea of putting things into families already appears in L -functions and its special value.

The second naturally occurring p -adic representations is the cyclotomic characters $\chi_p : G_F \rightarrow \mathbb{Z}_p^\times \rightarrow \overline{\mathbb{Q}_p}^\times$ (this shows up when we have construction of algebraic varieties that depends on roots of unity).

The third example is that of abelian varieties. Fix any prime $p > 0$ we can consider the p^r -power torsion point $A_{\overline{F}}[p^r]$. We can consider the action of the absolute Galois group on the torsion points, and take the inverse limit over all r and get the Tate module $T_p A_{\overline{F}}$ which is a free \mathbb{Z}_p -module of rank $2d$ as a \mathbb{Z}_p -module, or if we don't like modules, we can consider the rational Tate module $T_p A_{\overline{F}} \otimes \mathbb{Q}_p$, a $2d$ -dimensional \mathbb{Q}_p -vector space.

We can define abelian varieties over general base fields, even abelian schemes over general base schemes. Then $T_p A_{\overline{k}} \cong H_1^{et}(A_{\overline{k}}, \mathbb{Z}_p)$ (\mathbb{Z}_p -dual). More generally, for X algebraic variety over F , we get a continuous action of G_F on $H_{et}^i(X_{\overline{F}}, \mathbb{Q}_p)$ (and also the compactly supported version). We also have theory for cohomology of etale local systems, or more generally etale constructible sheaves, and finally take inverse limit and build \mathbb{Z}_p -cohomology and \mathbb{Q}_p -cohomology (one need to be careful about higher \varprojlim^i). The Tate twist is defined by $\gamma(j) := \gamma \otimes \chi_p^j$.

If we vary p , we get natural compatible system of representations. One thing we need to figure out is what compatibility means (c.f. the big conjecture).

One more Baire flavour argument: Suppose $\rho : \Gamma \rightarrow GL_n(\overline{\mathbb{Q}_p})$ factors through $:\rho := \Gamma \rightarrow GL_n(E)$. There exists a full lattice $L \subseteq E^n$ stabilized by $\rho(\Gamma)$. The proof is consider the standard lattice $\mathcal{O}_E^n \subseteq E^n$. This has open compact stabilizer $GL_n(\mathcal{O}_E)$. Take $\Gamma_0 = \rho^{-1}(GL_n(\mathcal{O}_E))$ open compact subgroup of Γ , so it has finite index inside Γ . Thus there exists finitely many elements g_1, \dots, g_r such that

$\Gamma = \sqcup g_i \Gamma_0$. We can take $L = \sum g_i L_0$. This allows us to analyse the representation by taking reduction mod p^m : $\rho_{L,m} : \Gamma \rightarrow \text{Aut}(L/p^m L)$ with finite image. The representation ρ can be thought of as $\varprojlim \rho_{L,m} \otimes_{\mathcal{O}_E} \overline{\mathbb{Q}_p}$.

The above examples are of this format. For the last one, look at the image of $H_{\text{et}}^i(X_{\overline{F}}, \mathbb{Z}_p)$ in $H_{\text{et}}^i(X_{\overline{F}}, \mathbb{Q}_p)$, which is $H_{\text{et}}^i(X_{\overline{F}}, \mathbb{Z}_p)/\text{torsion}$. We can consider reduction mod p -powers.

For fun, here is a [post](#) on quadratic reciprocity in the settings of function fields.