Automorphic lifting lecture 4-6

J'ignore • 11 Sep 2025

Some notation: $\mathbb{A}^\infty = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$ (upper script means away from a certain place); $\mathbb{A} = \mathbb{R} \times \mathbb{A}^\infty \cong \prod' \mathbb{Q}_v$; For general number field F, we define $\mathbb{A}_F = \mathbb{A} \otimes_{\mathbb{Q}} F \cong \prod' F_v$, and $G = Res_{F/\mathbb{Q}} GL_n$, the Weil restriction of GL_n from F to $\mathbb{Q}$ (representable by choosing generators and relation of F over $\mathbb{Q}$, but it is not true in general, see this post; The original reference is Grothendieck's Bourbaki lecture no. 221 (May 1961), Techniques … IV: les schemas de Hilbert, §4 c. Note that Grothendieck denotes the Weil restriction by). We equip $GL_n(\mathbb{A}_F)$ by viewing it as a subset of $\mathbb{A}_F^{n^2+1}$ (last coordinate being $1/\det$). For why we topologize it this way see this.

Exercise: Check the two topologies on ideles agree. (r is close to r' and r^{-1} is close to r'^{-1}, the problem are there are non-units that converge to 1)

Define the ideles class group $\mathcal{C}_F = F^\times \setminus \mathbb{A}_F^\times$; the group of fractional ideals $\mathcal{J}_F := \mathbb{A}_F^{\infty,\times} / \prod_{v\nmid\infty} \mathcal{O}_{F_v}^\times \cong \oplus_{v\nmid\infty} (F_v^\times / \mathcal{O}_{F_v}^\times)$. As an abstract group this is $\oplus \varpi_v^{\mathbb{Z}}$, infinite direct sum of $\mathbb{Z}$ with indexing set equal to all finite places. The ideal class group $Cl_F = F^\times \setminus \mathcal{J}_F = F^\times \setminus \mathbb{A}_F^{\infty,\times} / \prod_{v\nmid\infty} \mathcal{O}_{F_v}^\times$. The relationships are shown in this diagram.

For each modulus $\mathfrak{m}$ we get a ray class group $Cl_F^{\mathfrak{m}}$ similar to the class group Cl_F. To define this first let $\mathcal{U}_{\mathfrak{m}} = \prod_{v\nmid\infty, r_v>0} (1 + \mathfrak{p}_v^{r_v} \mathcal{O}_{F_v}) \times \prod_{v\nmid\infty} \mathcal{O}_{F_v}^\times$. Such $\mathcal{U}_{\mathfrak{m}}$ are cofinal among all open compact subgroups. At infinity places consider $G(\mathbb{R})^\circ$, which is product of copies of $\mathbb{R}_{>0}$ and $\mathbb{C}^\times$. Finally let $Cl_F^{\mathfrak{m}} = G(\mathbb{Q}) \setminus G(\mathbb{A}) / G(\mathbb{R})^\circ \mathcal{U}_{\mathfrak{m}}$.

Close finite index subgroups (same as open compact if we restrict to norm-one element of the adeles) of $\mathcal{C}_F$ are exactly those containing for some $\mathfrak{m}$ as above. More generally for each open compact $U \subseteq G(\widehat{Z})$, it is still true that $G(\mathbb{Q}) \setminus G(\mathbb{A}) / G(\mathbb{R})^\circ U$ is a finite set (weak approximation), and it is a group when G is commutative. If K/F is a finite Galois extension, then we get a norm map $N_{K/F} : \mathbb{A}_K^\times \to \mathbb{A}_F^\times$, which induces a map $\mathcal{C}_K \to \mathcal{C}_F$. Note that the image is an open subgroup of finite index.

Statement of Artin reciprocity:

- There exists canonical global Artin reciprocity map or norm residue symbol: $(, K/F) : G(\mathbb{A}) = \mathcal{C}_F = F^\times \setminus \mathbb{A}_F^\times \to Gal(K/F)^{ab}$ (topological abelianization if K/F is not finite, which is the maximal abelian Hausdorff quotient) and it is surjective (specific to number fields) with kernel or $N_{K/F}\mathcal{C}_K$.

- Existence of class fields: The assignment $K \mapsto N_K = N_{K/F}(C_K)$ defines a bijection from abelian extension K/F to closed subgroup of finite index N of C_F. For example $N = \mathcal{C}_F^{\mathfrak{m}}$ corresponds to the ray class field mod $\mathfrak{m}$.

- Local-global compatibility: For each place v of F with algebraic closure $\overline{F_v}$ of F_v extending $\overline{F}$, there is also a canonical local Artin reciprocity map $Art_{F_v} : F_v^\times \to Gal(\overline{F_v}/F_v)^{ab}$ with same existence of finite abelian extension of F_v for each open subgroup of finite index such that the diagram commutes.

A more natural formulation is to use the Weil group. This is done in Tate's Corvallis paper 'On Number-theoretic background'. Some remark:

- The weil group W_F satisfies some natural axioms. For example, we should have an isomorphism (as topological group) $W_F^{ab} \cong \mathcal{C}_F$ if F is global and $F^\times$ if F is local. We also should be able to write W_F as a projective limit $W_F \cong \varprojlim W_{K/F}$ where $W_{K/F} := W_F/W_K^c$. If F is local, W_F is coming from the diagram. To introduce a filtration on W_F, we consider $G_F \to Gal(K^{ab}/F) \to \widehat{\mathbb{Z}}$. The inverse image of $\mathbb{Z}$ in G_F is W_F, and define $W_{K/F} :=$ inverse image of $\mathbb{Z}$ in $Gal(K^{ab}/F)$. Note that $Gal(K^{ab}/F)$ surjects onto $Gal(K/F)$ with kernel $Gal(K^{ab}/K)$. Thus $W_{K/F}$ is an extension of $Gal(K/F) \cong W_F/W_K$ by W_K^{ab}, and from this it is not hard to check that it is isomorphic to W_F/W_K^c (This viewpoint also suggests that we can use galois cohomology to construct $W_{K/F}$ and therefore W_F).

- If K is local and $a \in K^\times$, then $Art_v^(a)|_{\overline{k}} = x \mapsto x^{|a|_K}$, so uniformizers correspond to inverse of the usual Frobneius automorphism $x \mapsto x^q$.

For a clean explanation of how Artin reciprocity implies quadratic reciprocity, see this wikipedia page. Essentially it has to do with how primes split in quadratic fields and cyclotimic fields and the compatibility of the Artin symbol.

Some Baire category arguments: Any continuous representation $\rho : \Gamma \to GL_n(\overline{\mathbb{Q}_p})$ where Γ is a compact topological group, factors through some $\Gamma \to GL_n(E_v)$ where E_v is a finite extension of $\mathbb{Q}_p$. The proof is to assume the following fact, the number of finite extensions E of $\mathbb{Q}_p$ is countable (since a finite extension E of $\mathbb{Q}_p$ is completion of a finite extension F over $\mathbb{Q}$, which is Krasner's lemma (algebraic extensions of nearby polynomials are equal)). For any such E, $GL_n(E) \subset GL_n(\overline{\mathbb{Q}_p})$ is closed, which implies $\Gamma = \cup_E \Gamma_E$ where $\Gamma_E = \rho^{-1}(GL_n(E))$, there exists E_1 such that Γ_{E_1} has positive Haar measure. Thus $\Gamma = \sqcup g\Gamma_{E_1}$ for finitely many disjoint union. Take E to be finite extension of E_1 given by entries of $\rho(g)$. In particular we can take Γ to be the absolute Galois group.

But why do we consider p-adic representation? Because they appear naturally. An Artin representation is a complex representation of the absolute Galois representation G_F that has finite image, i.e. factors through $Gal(K/F)$ for some finite extension K/F. In this case, once we fix an isomorphism between $\overline{\mathbb{Q}_p} \xrightarrow{\cong} \mathbb{C}$, we get a p-adic representation. The reason why this could be better is that we could put Artin representations into (p-adic) families, whereas it is much harder to take a complex families of Artin representations. The idea of putting things into families already appears in L-functions and its special value.

The second naturally occuring p-adic representations is the cyclotimc characters $\chi_p : G_F \to \mathbb{Z}_p^\times \to \overline{\mathbb{Q}_p}^\times$ (this shows up when we have construction of algebraic varieties that depends on roots of unity).

The third example is that of abelian varieties. Fix any prime $p > 0$ we can consider the p^r-power torsion point $A_{\overline{F}}[p^r]$. We can consider the action of the absolute Galois group on the torsion points, and take the inverse limit over all r and get the Tate module $T_p A_{\overline{F}}$ which is a free $\mathbb{Z}_p$-module of rank $2d$ as a $\mathbb{Z}_p$-module, or if we don't like modules, we can consider the rational Tate module $T_p A\overline{F} \otimes \mathbb{Q}_p$, a $2d$-dimensional $\mathbb{Q}_p$-vector space.

We can define abelian varieties over general base fields, even abelian schemes over general base schemes. Then $T_p A_{\overline{k}} \cong H_1^{et}(A_{\overline{k}}, \mathbb{Z}_p)$ ($\mathbb{Z}_p$-dual). More generally, for X algebraic variety over F, we get a continuous action of G_F on $H^i_{et}(X_{\overline{F}}, \mathbb{Q}_p)$ (and also the compactly supported version). We also have theory for cohomology of etale local systems, or more generally etale constructible sheaves, and finally take inverse limit and build $\mathbb{Z}_p$-cohomology and $\mathbb{Q}_p$-cohomology (one need to be careful about higher $\varprojlim^i$. The Tate twist is defined by $\gamma(j) := \gamma \otimes \chi_p^j$.

If we vary p, we get natural compatible system of representations. One thing we need to figure out is what compatibility means (c.f. the big conjecture).

One more Baire flavour argument: Suppose $\rho : \Gamma \to GL_n(\overline{Q_p})$ factors through $:_0 := \Gamma \to GL_n(E)$. There exists a full lattice $L \subseteq E^n$ stablized by $\rho(\Gamma)$. The proof is consider the standard lattice $\mathcal{O}_E^n \subseteq E^n$. This has open compact stablizer $GL_n(\mathcal{O}_E)$. Take $\Gamma_0 = \rho^{-1}(GL_n(\mathcal{O}_E))$ open compact subgroup of Γ, so it has finite index inside Γ. Thus there exists finitely many elements $g_1, ...g_r$ such that

$\Gamma = \sqcup g_i \Gamma_0$. We can take $L = \sum g_i L_0$. This allows us to analyse the representation by taking reduction mod p^m: $\rho_{L,m} : \Gamma \to Aut(L/p^m L)$ with finite image. The representation ρ can be thought of as $\varprojlim \rho_{L,m} \otimes_{\mathcal{O}_E} \overline{\mathbb{Q}_p}$.

The above examples are of this format. For the last one, look at the image of $H^i_{et}(X_{\overline{F}}, \mathbb{Z}_p)$ in $H^i_{et}(X_{\overline{F}}, \mathbb{Q}_p)$, which is $H^i_{et}(X_{\overline{F}}, \mathbb{Z}_p)/torsion$. We can consider reduction mod p-powers.

For fun, here is a post on quadratic reciprocity in the settings of function fields.