Abelian Galois representation implies potentially good reduction

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First we need a criterion for potentially good reduction in terms of Galois representation. The following is Theorem 2 from Serre & Tate's paper *Good Reduction of Abelian Varieties*:

> The abelian variety A has potential good reduction at a place v of K if and only if the image by $p : G_K \to T_\ell(A)$ of the inertia group $I(v)$ is finite.

Now the main result we aim to prove is the following (Corollary 1):

> Suppose that the residue field k is finite of characteristic p, and that, for some $\ell \neq p$, the image of G_K in $\mathrm{Aut}(T_\ell(A))$ is abelian. Then A has potential good reduction at v.

First we can complete K WLOG (since the Tate module of A is isomorphic to that of A' where A' is base change of A to the completion of K at v so if A' has potentially good reduction at v so is A by the above criterion). Local class field theory tells us that the image of inertia is actually a quotient of the group of units U_K of K. But structure theorem of unit groups of local fields tells us that U_K is the product of a finite group and a pro-p group. Thus the image intersect the pro-ℓ group $1 + \ell \cdot \mathrm{End}(T_\ell)$ trivially. Hence the image inject into $\mathrm{Aut}(T_\ell/\ell T_\ell)$, which is finite.