Obstruction theory

J'ignore • 3 Sep 2025

We would like to understand homotopy classes of maps between spaces. Recall an important construction, the mapping cylinder of $i : A \to X :=$ $X \bigsqcup (A \times I)/i(a) \sim (a, 0)$. Equivalently, it is pushout of $A \to A \times I$ (sending a to $(a, 0)$, inclusion) and $i : A \to X$; Note that A is a deformation retract of $A \times I$, and $M(i)$. Have a closed inclusion $A \to A \times I$ sending $a \mapsto (a, 1)$, and there exists an open neighborhood U of A such that U deformation retracts onto A. Finally, we can factor an arbitrary map i into closed inclusion into $M(i)$ followed by homotopy equivalence.

Now $Map(M(f), Y) \cong Map(X, Y) \times_{Map(A,Y)} Map(A \times I, Y)$. If we ask that the other copy of A (i.e. $A \times 1$) sent to a point y, the data amounts to a map $g : X \to Y$ and a homotopy from $g \circ f \cong c_y$ (null-homotopy), this amounts to a continuous map from the mapping cone $C(f)$ into Y.

Given a map $g : X \to Y$, we can extend it to $C(f)$ iff $g \circ f$ is nullhomotopic; $Map(C(f), Y) \to Map(X, Y)$ has fiber; A CW complex X is a sequence of spaces $X^{[i]}$ (inductive limit) where $X^{[0]}$ is a discrete set of points; $X^{[i]}$ is obtained from $X^{[i-1]}$ by forming pushout with $\bigsqcup D^i$ with attaching maps $\theta_i : \bigsqcup S^{i-1} \to X^{[i-1]}$.

Understanding in terms of mapping cones: $Map(X^{[i]}, Y)$ is a map $g_i : X^{[i-1]} \to Y$ and for each copy of S^{i-1} and homotopy $g \circ \theta_i|_{S_n^{i-1}} \cong c_{y_i}$ (the image of the orgin in D^i.)

Conclusion: $Map(X, Y)$ is inverse limit of $Map(X^{[i]}, Y)$. For $i = 0$, $Map(X^{[0]}, Y) = \prod_{X^{[0]}} Y$.

Note that $f \circ \theta_i|_{S_n^{i-1}} : S^{i-1} \to Y$ is an element of $\pi_{i-1}(Y)$. We get a function $I_i \to \pi_{i-1}(Y)$, extending to a homomorphism from the free abelian group $\mathbb{Z} \cdot I_i$ (cellular cochain). The map is extensible iff this cochain is zero, which is not a homological condition. However it is easy to prove two things:

1. This cochain is a cocycle.
2. If f is homotopic to f', then their corresponding cochains (cocycles by (1)) differ by a coboundary.

Thus if the cohomology class vanishes, we can choose f' homotopic to f such that f' can be extended. We might worry about that there is a tree of possibilities if we increase i. The miraculous thing about obstruction theory is that this is not the case.