What comes before Deligne Lusztig theory

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Before Deligne-Lusztig theory, we know very little about representations of finite groups of Lie type. The case of $GL_n(\mathbb{F}_q)$ is worked out in a paper by J. A. Green in 1955. Then the case of $Sp_4(\mathbb{F}_q)$ is done by Bhama Srinivasan, which provides the first example of unipotent cuspidal representations. Then in 1974, B. Chang and R. Ree figured out the case of $G_2(\mathbb{F}_q)$. That's virtually all of what we knew prior to the ground-breaking paper by Deligne-Lusztig in 1976 using geometric methods to study representations of general reductive groups. The contemporary thread is that representation <- geometry <- combinatorics. Deligne-Lusztig theory provides the first link, while the works of Kazhdan-Lusztig tells us we can use the Weyl group or Hecke algebra to study the geometry of Deligne-Lusztig varieties.

What we know before is that from a character θ of max torus (maximal abelian diagonalizable subgroups) $T \subset GL_2(\mathbb{F}_q)$ we get a representation of $GL_2(\mathbb{F}_q)$ by induction R_T^θ. The problem is that maximal tori over $\mathbb{F}_q$ are not all isomorphic. What Deligne-Lusztig theory tells us is how to do induction from non-split tori.

Let's review the classical theory of induction. For what follows, let $G = SL_2$, T be the maximal torus of diagonal matrices, B the Borel subgroup of upper triangular matrices, and U the unipotent radical of upper triangular matrices with all ones on the diagonal. We start with the case of $\mathbb{C}$. Let θ_n be the character of T given by

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto a^n,$$

let V_n be the space of holomorphic functions $f : SL_2(\mathbb{C}) \to \mathbb{C}$ such that f is holomorphic and $f(bg) = \theta_n(b) f(g)$ for $b \in B$. This is the same as function $U \setminus SL_2(\mathbb{C}) \to \mathbb{C} \cong \mathbb{A}^2 \setminus \{0\} \to \mathbb{C}$. Hartog's theorem tells us that f extends to a holomorphic function $\tilde{f} : \mathbb{A}^2 \to \mathbb{C}$. such that $\tilde{f}(ac, ad) = a^n \tilde{f}(c, d)$, which implies $\tilde{f}$ is homogeneous polynomial of degree $n \geq 0$, so $V_n \cong Sym^n(\mathbb{C}^2)$ with $\dim V_n = n + 1$.

Similarly for $GL_2(\mathbb{F}_q)$, we start with a character $\theta = (\theta_1, \theta_2)$ where $\theta_i : \mathbb{F}_q^\times \to \mathbb{C}$ (there are $q - 1$ many of them), define $R_T^\theta := Ind_B^G \theta$ to be essentially the above construction.

We next compute how many irreducible representation arises from this construction and how many are missing. We start with computing the number of irreducible representations, which is the number of conjugacy classes. We claim there are q^2-1 many of them. Basic combinatorics gives that $|GL_2| = (q^2-1)(q^2-q)$ and $|B| = (q-1)^2 q$, $|T| = (q-1)^2$, $\dim R_T^\theta = |G|/|B| = q+1$. The conjugacy classes in GL_2 can be classified by characteristic polynomials. There are three cases, either it is a complete square over $\mathbb{F}_q[X]$ (two choices for each $q \in \mathbb{F}_q^\times$; or it has distinct eigenvalues (for each unordered pair $\{a,b\}$, $a,b \in \mathbb{F}_q^\times$; or if it is irreducible of the form x^2+ax+b, in which case it is conjugate to $\begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}$ (number = $q^2-q-q(q-1)/2$).

Below are three facts about the representations R_T^θ:

1. If $\theta_1 \neq \theta_2$, then R_T^θ is irreducible;
2. $R_T^\theta \cong R_T^{\theta'}$ iff $\theta = \theta'$ or θ differs from θ' by a flip ;
3. $R_T^1 = 1 \oplus St$ where St is a q-dimensional irreducible representation (Steiberg representation);
4. If $\theta_1 = \theta_2 = \chi$, then $R_T^\theta \cong (\chi \circ \det) \oplus (\chi \circ \det \otimes St)$.

Finally the counting gives $(q-1)(q-2)/2 + 2(q-1) = (q^2+q-2)/2$ many irreducible representations from R_T^θ, and we need $(q^2-q)/2$ missing ones.

Now we prove the facts we just used. The three main ingredients are

- Mackey's formula: $Res_{H'}^G Ind_H^G \rho = \bigoplus_{w \in H\backslash G/H'} Ind_{w^{-1}Hw \cap H'}^{H'} \rho^w$.
- Projection formula: $Ind_H^G(\rho \otimes Res_H^G \rho') = (Ind_H^G \rho) \otimes \rho'$
- Bruhat decomposition: $W = T \setminus N(T)/T \to B \setminus GL_2/B$ is an isomorphism

From this we easily get the lemma $\dim \mathrm{End}(Ind_B^G \theta) = |\{w \in W : \theta^w = \theta'\}|$ by applying Frobenius reciprocity and Mackey's formula.

Using this (and the projection formula for 4) we easily prove all four facts.

There is a map $\mathbb{F}_{q^2}^\times \to GL_2(\mathbb{F}_q)$ (depending on a choice of $\tau \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. The image is an example of nonsplit torus. Note that τ has minimal polynomial $\tau^2 + a\tau + b = 0$ where $a = -(\tau + \tau^q)$ and $b = \tau\tau^q = \tau^{q+1}$. From this we see that in matrix form $\phi(\tau) = \begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}$; Also, over $GL_2(\mathbb{F}_{q^2})$, $\phi(g)$ is conjugate to $\begin{pmatrix} g & 0 \\ 0 & g^q \end{pmatrix}$

Let T_s stands for the nonsplit torus (s stands for the nontrivial Weyl group element). This is something more general: Deligne-Lusztig showed that the $G(\mathbb{F}_q)$-conjugacy classes of max torus in $G(\mathbb{F}_q)$ are in bijection with conjugacy classes in the Weyl group.

There are (q^2-1) many characters of T_s, but some of them come from the split torus from the norm map $Nm : \mathbb{F}_{q^2} \to \mathbb{F}_q$ which is $x \mapsto x^{q+1}$. This map is surjective. Therefore a character χ of $\mathbb{F}_q^\times$ pullback to a character θ of $\mathbb{F}_{q^2}$. The regular characters are those that don't come from this, whose number is $(q^2-1)-(q-1)=q^2-q$. An easy criterion for regular character is that $\theta \neq \theta^q$ (essentially because of Hilbert 90, since $\theta(kerNm)=1$ iff $\theta(y^q)=\theta(y)$). The number of orbits of the set of regular characters under F^* has number is $(q^2-q)/2$.

Algebraic construction (due to Gelfand-Graev): For each regular $\theta : T_s \to \mathbb{C}$, we want to construct an irregular representation of dimension $R^\theta_{T_s}$ of dimension equal to $q-1$, and $R^\theta_{T_s} = R^{\theta q}_{T_s}$. Consider the nontrivial character $\psi : \mathbb{F}_q \cong U \to \mathbb{C}^\times$ given by $u \mapsto \exp(\frac{2\pi i \mathrm{Tr}(u)}{p})$ (or just pick one). Consider $\Gamma_\psi = Ind_U^{GL_2}\psi$ (called the G-G representation). Consider the isotypic component $\Gamma_{\psi,\theta} = Ind^G_{ZU}(\theta|_Z \boxtimes \psi)$. Consider $R^\theta_{T_s} = \Gamma_{\psi,\theta} - Ind^G_{T_s}\theta$. The proof that this is the sought-after cuspidal representations is just to compute everything. More precisely, let χ be the character (virtual a priori), we need five computations (using formula for character of induction):

- $\chi(\begin{pmatrix} z & \\ & z \end{pmatrix}) = (q-1)\theta(z)$, $z \in \mathbb{F}_q^\times$
- $\chi(\begin{pmatrix} z & \\ & z \end{pmatrix}\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}) = -\theta(z)$, $0 \neq u \in \mathbb{F}_q$
- $\chi(\begin{pmatrix} a & \\ & b \end{pmatrix}) = 0$ if $a \neq b, a, b \in \mathbb{F}_q$
- $\chi(z) = -(\theta(z) + \theta^q(z))$ if $z \in T_s \setminus Z \cong \mathbb{F}_{q^2} \setminus \mathbb{F}_q$
- $\langle \chi, \chi \rangle = 1$

Note that $\chi(id) = q-1 > 0$, so $R^\theta_{T_s}$ is an irreducible representation.

Note that GG-representation Γ_ψ makes sense for general $G(\mathbb{F}_q)$, all we need to replace is U by the unipotent radical of a split Borel and a nontrivial character ψ. In the previous case, we see all cuspidal representaions are summands of Γ_ψ. This is true for GL_n but fails for Sp_4, so we need some other way for general groups of Lie type.

Fact about Γ_ψ: Multiplcity one holds for Γ_ψ. In the case of $GL_2(\mathbb{F}_q)$, all irreps except the one-dimensional one appears. Irreducible summands of Γ_ψ are generic (i.e. admits Whittaker model).

Another exercise is compute what is $Ind_{T_s}^G\theta$, it is harder to use Mackey formula since it doesn't extend to Borel so we don't have the Bruhat decomposition.

Geometric construction: Consider $X_s = V(x^q y - y^q x \in \mathbb{F}_q^\times) \subseteq \overline{\mathbb{F}_q}^2$, both $GL_2(\mathbb{F}_q)$ and T_s act on X_s and their actions commute, and if we consider the induced actions on $H^1_{et}(X_s)$ then this gives us a way to match the representations of $GL_2(\mathbb{F}_q)$ with the characters of T_s.

Weil representation: $W_\theta = \{f : \mathbb{F}_{q^2} \to \mathbb{C} : f(yx) = \theta(y)^{-1} f(x), \forall y \in kerNm\}$. If θ is regular, then $\dim W_\theta = q - 1 = \frac{q^2-1}{q+1}$. Andre Weil discovered that this space carries an action of GL_2:

- $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} f(x) = f(ax)$
- $\begin{pmatrix} a & \\ & 1 \end{pmatrix} f(x) = \theta(b) f(bx)$, $Nm(b) = a$
- $\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} f(x) = \psi(uNm(x)) f(x)$
- $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} f(x) = \widehat{f}(x) = (-q^{-1}) \sum_{y \in \mathbb{F}_{q^2}} f(y)\psi(tr(x^q y))$

where $\psi : \mathbb{F}_q \to \mathbb{C}$ is a nontrivial character of the additive group.