p-divisible groups and what they are for

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We know for elliptic curves E over a field of characteristic p, the ℓ-adic Tate module $T_\ell E$ determines the isogeny class of the elliptic curve for $\ell \neq p$. However, if we want to study deformations or variation in families, then using $T_\ell E$ is inadequate since the ℓ-adic and the p-adic topologies are incompatible. On the other hand, the p-adic Tate module $T_p E$ doesn't contain enough information (whose rank is either 0 or 1 depending on whether E is ordinary or supersingular). The solution is to consider $E[p^\infty]$ as a p-divisible group. We have the following important result:

> (Serre-Tate) Given E/k, the category of deformations of E is naturally equivalent to the category of deformations of $E[p^\infty]$.

The result holds more generally for abelian varieties. The intuition is that taking the Tate module is insufficient because it only looks at the geometric point of the finite flat group scheme of $E[p^n]$. In the case of $E[\ell^n]$ this is enough since it is etale. Thus the idea is to look at the inductive system of finite flat group schemes $\{E[p^n]\}$.

There are lots of great introductions to this topic, e.g. this blog post by Alex Youcis (although note that there are some errors/typos, espeically in the preliminary materials on finite flat group schemes; for this see this lecture note by Richard Pink). One important result is that a p-divisible group over a complete Noetherian local ring is entirely determined by its generic fiber, more precisely:

> Let R be a complete Noetherian local domain with residue characteristic p, and $K := \mathrm{Frac}(R)$ of characteristic 0. Then, the functor
>
> $$\mathrm{BT}_p(R) \to \mathrm{BT}_p(K) : G \mapsto G_K$$
>
> is fully faithful, where $BT_p(R)$ denotes the category of p-divisible groups over R. This is very useful, since the cateogry of p-divisible groups over the field K is just that of $\mathbb{Z}_p$-modules with action of the absolute Galois group $Gal(\overline{K}/K)$.

This is the main result of Tate's 1967 paper on p-divisible groups. We will discuss it next.