Exercises on Algebraic Geometry

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This post contains qualifying exam questions on algebraic geometry.

- 1. Let C be a smooth complex projective curve of genus g. For any $P \in C$, prove the following statements:
- a. For any $k \geq 2g$, there is always a nonconstant rational function on C which is regular everywhere except for a pole of order k at P. (By Riemann-Roch we have $\dim H^0(C, \mathcal{O}_C(kP)) = k g + 1$ for $k \geq 2g 1$).
- b. (Weierstrass gaps) There are exactly g numbers $0 < k_1 < k_2 < ... < k_g < 2g$ such that for each $1 \le i \le g$, there is no nonconstant rational function on C which is regular everywhere except for a pole of order k_i at P. (The 2g numbers $\{\dim H^0(C, \mathcal{O}_C(kP) : 0 \le k \le 2g-1\}$ are between 1 and g and by Riemann-Roch each increment is at most 1.)

2.

- a. Let $X \subset \mathbb{P}^n$ be a Zariski closed subset. Define the Hilbert function $h_X(m)$ and the Hilbert polynomial $p_X(m)$. (Let I(X) be the defining homogeneous ideal of X; then $h_X(m)$ can be defined as the codimension of the m-th graded piece. For large values of m this is a polynomial in m, called the Hilbert polynomial of X.)
- b. Suppose $X = \{p_1, ..., p_d\} \subset \mathbb{P}^n$. Show that $h_X(d-1) = d$. (It suffices that for any $1 \le i \le d$, we find a homogeneous polynomial of degree d-1 that vanishes at all p_j except p_i . We simply take product of d-1 linear forms that vanishes at p_j but non-zero at all other points.)
- c. Again, suppose $X = \{p_1, ..., p_d\} \subset \mathbb{P}^n$. Show that $h_X(d-2) = d$ unless X is contained in a line. (Again as in (b) but this time we need to find a homogeneous polynomial of degree d-2 that vanishes at all but one point. The idea is to if not all points are colinear, then for any i, we can

- find j, k and a linear form L that vanishes at p_j and p_k but not p_i . Then we simply take the product of L with d-3 other linear forms that vanishes at p_l for $l \neq i, j, k$.)
- 3. Let $X \subset \mathbb{P}^n$ be a variety of dimension k, let G(1,n) be the Grassmannian parametrizing lines in \mathbb{P}^n and let $F_1(X) := \{L \in G(1,n) | L \subset X\}$ be the locus of lines contained in X. Show that $\dim F_1(X) \leq 2k-2$ with equality holding only if X is a k-plane in \mathbb{P}^n . (Consider the map $(X \times X) \setminus \Delta \to G(1,n)$ sending $(p,q) \mapsto l_{pq}$ the line connecting p and q and comparing dimension.)
- 4. Show that Grassmannian is a proper scheme. (use the representability criterion, see here for answer; for properness, use the exterior power embedding into projective space and show that the image consisting of rank-one tensors is closed, and it represents the Grassmannian functor. This is in fact a closed immersion, see here. The idea is that if we put a $d \times n$ -matrix containing a $d \times d$ identity submatrix, then the set of $d \times d$ -minor determines the remaining $d \times (n-d)$ entries.)
- 5. For a commutative affine algebraic group G, show that G_s and G_u are closed subgroups and there is an isomorphism $G_s \times G_u \to G$. (The fact that they are subgroups follow from embedding G into GL_n and uses the fact that commuting elements can be simultaneously diagonalized. The closedness of G_s follows from being constructible. For more details see here. The fact that this is an isomorphism follows from bijectivity, see here)
- 6. Assume char(k) = p > 0. Let $g \in G$ be an element. Show that g is unipotent iff $g^{p^n} = e$ for some $n \ge 1$ and g is semisimple iff $g^{p^n-1} = e$ for some $n \ge 1$ (For unipotent this is easy; for semisimplicity note that every eigenvalue of g lies in some finite field. This can be used to show Jordan decomposition in this case).