Exercises on Algebraic Geometry

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This post contains qualifying exam questions on algebraic geometry.

1. Let C be a smooth complex projective curve of genus g. For any $P \in C$, prove the following statements:

a. For any $k \geq 2g$, there is always a nonconstant rational function on C which is regular everywhere except for a pole of order k at P. (By Riemann-Roch we have $\dim H^0(C, \mathcal{O}_C(kP)) = k - g + 1$ for $k \geq 2g - 1$).

b. (Weierstrass gaps) There are exactly g numbers $0 < k_1 < k_2 < ... < k_g < 2g$ such that for each $1 \leq i \leq g$, there is no nonconstant rational function on C which is regular everywhere except for a pole of order k_i at P. (The $2g$ numbers $\{\dim H^0(C, \mathcal{O}_C(kP) : 0 \leq k \leq 2g - 1\}$ are between 1 and g and by Riemann-Roch each increment is at most 1.)

2.

a. Let $X \subset \mathbb{P}^n$ be a Zariski closed subset. Define the Hilbert function $h_X(m)$ and the Hilbert polynomial $p_X(m)$. (Let $I(X)$ be the defining homogeneous ideal of X; then $h_X(m)$ can be defined as the codimension of the m-th graded piece. For large values of m this is a polynomial in m, called the Hilbert polynomial of X.)

b. Suppose $X = \{p_1, ..., p_d\} \subset \mathbb{P}^n$. Show that $h_X(d-1) = d$. (It suffices that for any $1 \leq i \leq d$, we find a homogeneous polynomial of degree $d - 1$ that vanishes at all p_j except p_i. We simply take product of $d - 1$ linear forms that vanishes at p_j but non-zero at all other points.)

c. Again, suppose $X = \{p_1, ..., p_d\} \subset \mathbb{P}^n$. Show that $h_X(d-2) = d$ unless X is contained in a line. (Again as in (b) but this time we need to find a homogeneous polynomial of degree $d - 2$ that vanishes at all but one point. The idea is to if not all points are colinear, then for any i, we can find j, k and a linear form L that vanishes at p_j and p_k but not p_i. Then we simply take the product of L with $d-3$ other linear forms that vanishes at p_l for $l \neq i, j, k$.)

3. Let $X \subset \mathbb{P}^n$ be a variety of dimension k, let $G(1,n)$ be the Grassmannian parametrizing lines in $\mathbb{P}^n$ and let $F_1(X) := \{L \in G(1,n) | L \subset X\}$ be the locus of lines contained in X. Show that $\dim F_1(X) \leq 2k-2$ with equality holding only if X is a k-plane in $\mathbb{P}^n$. (Consider the map $(X \times X) \setminus \Delta \to G(1,n)$ sending $(p,q) \mapsto l_{pq}$ the line connecting p and q and comparing dimension.)