Qualifying exam questions: Complex Analysis

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This post contains qualifying exam questions on complex analysis.

1. Prove Schwarz's lemma: Let $D \subseteq \mathbb{C}$ be the open unit disk $f : D \to D$ such that $f(0) = 0$ holomorphic function. Show that $|f'(0)| \le 1$. (Use Cauchy's integral formula, we get that $\frac{1}{2\pi i}\oint_{|w|=r} \frac{f(w)}{w^2} dw = f'(0)$ for each $r < 1$.)

2. Suppose $u \notin \mathbb{Z}$. Show that $\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{\sin^2 \pi u}$.(Use residue method. Consider the function $f(z) = \frac{\pi \cot \pi z}{(u+z)^2}$ and the contour integral $\oint_{|z|=N+\frac{1}{2}} f(z)dz$ as $N \to \infty$. This has a single pole at every integers n and a double pole at $z = -u$. The former contributes $(u+n)^{-2}$ to the residue while the latter contributes $\pi \frac{d}{dz}|_{z=u} \cot(\pi z) = -\frac{\pi^2}{\sin^2 \pi u}$. Note that as $N \to \infty$ the function $\cot(\pi z)$ on the circle $|z| = N + \frac{1}{2}$ is bounded.)

Note: The same method works to compute when $u = 0$. Alternatively, we can compute $\lim_{u\to 0} \frac{\pi^2}{\sin^2 \pi u} - \frac{1}{u^2}$.

3. Suppose f, g are holomorphic on a neighborhood of the closed unit disc in $\mathbb{C}$. Suppose that $f(z)$ has a simple zero at $z = 0$ and vanishes nowhere else on the closed unit disc. For any $\epsilon > 0$, define $f_\epsilon(z) = f(z) + \epsilon g(z)$.

(a) Show that if $\epsilon > 0$ is small enough, then f_ϵ has a unique zero in the closed unit disc.

(b) Show that if $\epsilon > 0$ is small enough and the unique zero from part (1) is denoted by z_ϵ, then $z_\epsilon \to 0$ as $\epsilon \to 0$ from above. (Use Rouché's theorem on the circle of radius δ for any $\delta > 0$.)

4. Determine the radius of convergence of the power series for $\sqrt{z}$ at $z_0 = -3 + 4i$. (The radius of convergence around a point is distance to the nearest singularity of the power series, see this answer for a proof which uses contour integral and Cauchy integral formula. Note that this is not the same as the radius of the circle for which the power series agrees with the principal branch of $\sqrt{z}$.)

5. Change variables on the elliptic curve $z^2 = w^4 - 1$ to write it in the approximately Weierstraß form $y^2 =$ cubic in t. (Recall the proof of the existence of Weierstraß form using Riemann-Roch is constructive; The key is to find rational functions with poles of order 2 and 3 at one of the puncture and regular elsewhere. Rewrite

$$z = \sqrt{w^4 - 1} = w^2\sqrt{1 - \frac{1}{w^4}} = w^2(1-(1/2)w^{-4}+...) = w^2-(1/2)w^{-2}+...$$

Thus the function $x := z - w^2$ is regular at the puncture $z \approx w^2$ and so is $y := w(z - w^2)$. On the other hand, they have a pole of order 2 and 3 near $z \approx w^2$. Plugging in $w = y/x$ and $z = x + (y/x)^2$ into the quartic, we get $x^2 + 2y^2/x + (y/x)^4 = (y/x)^4 - 1$, which is equivlanet to $-2y^2 = x^3 + x$.)