Qualifying exam question (Real Analysis, Measure Theory, Functional Analysis)

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This is the third post on qualifying exam preparation and it will be on questions in real analysis, measure theory, and some functional analysis.

1. Let $L^2([1,2])$ be the Hilbert space of real-valued integrable functions with inner product. Consider the linear operator given by $Tf(x) := xf(x)$.

a. Show that this map is continuous and invertible (with continuous inverse), and show that T is self-adjoint.

b. Show that T has no (non-zero) eigenvectors.

c. Fix any $\lambda \in [1,2]$. Find a sequence $(f_n) \subset L^2([1,2])$ such that $\|f_n\|_{L^2([1,2])} = 1$ for all n and $\|(T - \lambda I)f_n\|_{L^2([1,2])} \to 0$.

(This question is essentially definition-checking; For (c) let f_n be supported on a interval of length $1/n$ containing λ.)

2. Given $\epsilon > 0$, exhibit an open subset of $\mathbb{R}$ containing every rational number and having Lebesgue measure less than ϵ. (Enumerate the rationals and take the union of intervals of length $\epsilon/2^n$ around them.)

3. Show that the closed unit ball in l^2 is not compact. (Let $f_n(x) = 1$ if $x = n$ and 0 otherwise. It is not Cauchy so cannot converge to anything.)

4. Let us define a topology τ on the real line $\mathbb{R}$ in the following way: By definition, a set $U \subset \mathbb{R}$ is $\tau-$open if an only if for each $x \in U$ there is a compact subset $K \subset U$ such that

$$\lim_{h \to 0^+} \frac{|(x-h, x+h) \cap K|}{2h} = 1,$$

where, for a Lebesgue-measurable set $E \subset \mathbb{R}$, we denote by $|E|$ its Lebesgue measure.

a. Verify that the definition actually gives a topology.

b. Show that any $\tau-$open set is Lebesgue measurable. (Recall a set $E \subset \mathbb{R}$ is Lebesgue measurable if it can be covered by an open set U such that the outer measure of $U \setminus E$ is arbitrarily small. Use the infinite version of the Vitali Covering Theorem.)

c. Is every $\tau-$open set a Borel set? (Take the complement of a measure zero subet that is not Borel measurable; note that Lebesgue measure is inner regular)

d. Is the real line connected in the $\tau-$topology? (Yes. Note that for fixed $\delta_1 > 0$ the function $f_{\delta_1}(x) := \frac{|(x-\delta_1, x+\delta_1) \cap U|}{2\delta_1}$ is continuous, and if we can disconnect $\mathbb{R}$ in the τ-topology, then f takes values close to 0 and close to 1. Thus we can find $x \in \mathbb{R}$ such that $f(x) = 1/2$. We can repeat the same argument with some $\delta_2 < \delta_1$, since the average of f_{δ_2} in $(x - (\delta_1 - \delta_2), x + (\delta_1 - \delta_2))$ is close to $1/2$ if δ_2 is sufficiently small. Taking $\delta_i \to 0$, we see that $\lim_{h \to 0^+} \frac{|(x-h, x+h) \cap K|}{2h}$ either doesn't exist or $\neq 0, 1$.)

5. Show that $\mathbb{Q}$ cannot be the set of points of continuity of a real-valued function $f : \mathbb{R} \to \mathbb{R}$. (Such set must be G_δ, i.e. a countable intersection of open subsets, but $\mathbb{Q}$ is not by Baire Category.)

6. Use double integral to compute $\int_0^\infty \frac{\sin x}{x}$. (The key identity is $\frac{1}{x} = \int_0^\infty e^{-xy} dy$. To justify the change of order of integration, we use Fubini-Torelli Theorem. See this answer for detail.)

7. By Holder's inequality, we have $L^2([a, b]) \subset L^1([a, b])$. Show that it is of first category. (Take $g_n := n$ for $x \in [0, 1/n^3]$ and 0 otherwise. Then for any $f \in L^2$, we have $fg_n \to 0$ in L^1 by Holder. Thus $L^2 \subset \bigcup B_n$ where $B_n := \{f \in L^1 : \|fg_n\|_{L^1} \leq 1\}$. Each B_n is closed and has empty interior, since we can construct a function $h \in L^1 \setminus L^2$ such that $\|h\|_{L^1} \leq \epsilon$ and $\|hg_n\|_{L^1} \geq 2$ for all n.)

8. Show that the analogue of invariance of domain for infinite-dimensional Banach space is false, i.e. find a Banach space X and a continuous injective self-map $f : X \to X$ such that the image is not open. (Take the right shift $l^\infty \to l^\infty$.)

9. Show that if $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, then (a) it preserves any measure zero susbets. (This is almost immediate by definition of absolute continuity, the only trick being that given a finite covering by intervals $\{(a_i, b_i)\}$ of total length less than ϵ, we can further refine it so that $f(I_i)$ is contained in an interval of length close to $|f(a_i) - f(b_i)|$ by continuity.) (b) it preserves measurable subsets. (Any Lebesgue measurable subset of $\mathbb{R}$ can be written as the disjoint union of a Borel measurable subset of the same measure and a measure zero subset by outer regularity. The image of

Borel subset under continuous functions, so called analytic set, is Lebesgue measurable. This is an entry-point of descriptive set theory, for a proof see this answer.)