Qualifying exam question (Algebraic topology)

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Following Qualifying exam question (Algebra, Representation theory, Category Theory and Homological Algebra), this post will discuss questions in Algebraic Topology.

1. Suppose X is a finite CW complex such that $\pi_1(X)$ is finite and nontrivial. Show that the universal covering $\tilde{X}$ of X cannot be contractible. (Note that $\chi(\tilde{X}) = |\pi_1(X)|\chi(X)$, in particular it cannot be 1.)

2. Let Σ_g denotes the closed orientable surface of genus g. There is a degree 1 map from Σ_g to Σ_h iff $g \geq h$. (If $g \geq h$, we can construct a degree one map by writing $\Sigma_g = \Sigma_h \# \Sigma_{g-h}$ and contracting Σ_{g-h} to a point; If $g < h$ and $f : \Sigma_g \to \Sigma_h$ is a degree 1 map, then $f^* : H^1(\Sigma_h) \to H^1(\Sigma_g)$ is not injective for dimensional reason. Let $x \neq 0$ be in the kernel, and by Poincare duality let $y \in H^1(\Sigma_h)$ such that $x \cup y$ generates $H^2(\Sigma_h)$. Then $f^*(x \cup y) = f^*(x) \cup f^*(y) = 0$. However, by the (cohomological) definition of degree, f^* induces an isomorphism of H^2, contradiction. This in fact shows that any continuous map from Σ_g to Σ_h has degree 0.)

Remark: See this question and this paper for what degree could arise when $g \geq h$.

3. What are the homology of $\mathbb{RP}^2 \times \mathbb{RP}^3$,

a. with coefficients in $\mathbb{Z}$?

b. with coefficients in $\mathbb{Z}/2$?

c. with coefficients in $\mathbb{Z}/3$?

(For (a) the fastest way to do this is via cellular homology, the cellular chain complex for $\mathbb{RP}^n$ is $0 \to \mathbb{Z} \to \mathbb{Z} \to ...\mathbb{Z} \to 0$ corresponding to one cell in each dimension; the gluing map is via $S^{k-1} \to \mathbb{RP}^{k-1} \to \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1}$; the computation of degree of this map can be done locally, and it is the sum of the degree of the identity map and the antipodal map, i.e. $1 + (-1)^k$. Therefore the map in the cellular chain complex is 0 and multiplication by 2 alternately. The answer is $\mathbb{Z}, \mathbb{Z}/2, 0, ..., 0$ or $\mathbb{Z}$ in ascending degrees. Then we can compute the integral cohomology of $\mathbb{RP}^2 \times \mathbb{RP}^3$ using Kunneth's formula, and we get

$\mathbb{Z}, (\mathbb{Z}/2)^2, \mathbb{Z}/2, \mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/2, 0$; For (b) and (c) we use universal coefficient theorem and get $\mathbb{Z}/2, (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^3, (\mathbb{Z}/2)^3, (\mathbb{Z}/2)^2, \mathbb{Z}/2$ and $\mathbb{Z}/3, 0, 0, \mathbb{Z}/3, 0, 0$.)

Remark: Don't forget the Tor_1 term when using Kunneth and Universal Coefficient.

4. Compute the cohomology $H^i(\mathbb{R}P^n, \mathbb{Z}/2)$ for all i and $n \geq 1$. (Use the universal coefficient theorem for Ext, the degree i cohomology $H^i(X; M)$ with coefficients in M surjects onto $\mathrm{Hom}_{\mathbb{Z}}(H_i(X;\mathbb{Z}), M)$ with kernel $\mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(X;\mathbb{Z}), M)$. The answer is $\mathbb{Z}/2$ for all $0 \leq i \leq n$ and 0 otherwise.)

5. Show that every closed, connected, compact (but not necessarily orientable) smooth manifold of odd dimension has Euler characteristic 0. (Suppose first the manifold is orientable, then we just use Poincaré duality $H^i(M, \mathbb{Z}) \cong H^*_{n-i}(M, \mathbb{Z})$ and note that $rk_{\mathbb{Z}} H^i = rk_{\mathbb{Z}} H_i$. For nonoroentable manifolds we can either consider its orientable double cover, or apply Poincaré duality with $\mathbb{Z}/2$ coefficients.)

6. Let T be a closed orientable surface of genus 2.

a. Prove the homotopy group $\pi_2(T) = 0$. (Its universal cover can be taken to be the hyperbolic plane, which is contractible, so has all homotopy groups zero. Note that any map of S^2 into T lifts to its universal cover.)

b. Show that $\pi_1(T)$ is nonabelian. (The fundamental group $\pi^1(T) = \langle a, b, c, d \mid [a, b][c, d] = 1\rangle$ admits a surjection onto the free group in two generators.)

7. Find the fundamental group of $\mathbb{R}^2$ minus two points. (Use van-Kampen theorem.)

8. Let X be a path connected, locally path connected, semi-locally simply connected space, and assume that

$$\pi_1(X, x) = \mathbb{Z}/2 \times \mathbb{Z}/3.$$

a. Classify all isomorphism types of path connected covering spaces of X. (By Galois correspondence of covering spaces, there is a bijection between subgroups of $\pi_1(X)$ and path-connected covering spaces of X.)

b. For each path connected covering space, describe the group of deck transformations. (The group of deck transformation is the automorphism group of the cover. It is equal to $N(H)/H$ if E is associated to the subgroup $H \subsetneq \pi_1(X)$.)

9. Compute the singular homology of the Klein bottle with $\mathbb{Z}/2$-coefficient. (The answer is $\mathbb{Z}/2, \mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2$ in degree $0, 1, 2$ and 0 otherwise, which is consistent with Poincare duality.)

10. Prove the theorem on invariance of dimension, i.e. two manifolds of different dimensions cannot be homeomorphic, by considering the relative homology $H_k(M, M \setminus \{m\})$ (Using excision, we can compute the relative chomology $H_k(M, M \setminus \{m\}) \cong H_k(K, K \setminus \{m\})$ where K is a compact neighborhood of m homeomorphic to the n-dimensional closed disk.)

Universal coefficient theorem proof: Use the exact sequence $0 \to Z_\bullet \to C_\bullet \to B_{\bullet-1}$. Take Hom (note that it remains exact since B_n is free abelian) and then take the LES in homology; On the other hand, we have a SES $0 \to B_\bullet \to Z_\bullet \to H_\bullet \to 0$ that is actually a free resolution of $H_\bullet$. The proof essentially identifies the two LES in homology. Note that cohomology is determined by homology, but it is not the other way around. See this question for a reference. The intuition is that cohomology is contravariant, so from the diagonal map $X \to X \times X$ we get a map $H^n(X \times X) \to H^n(X)$. From Eilenberg-Zilber (right to left) and Alexander-Whitney (left to right) we know that $C_\bullet(X \times Y) \xrightarrow{\cong} C_\bullet(X) \otimes C_\bullet(Y)$. However, for the (co)homology of tensor product of complexes, we only have the following SES:

$$0 \to H_*(C) \otimes H_*(D) \to H_*(C \otimes D) \to \mathrm{Tor}(H_*(C), H_*(D)) \to 0,$$

this sequence splits but not naturally. In any case the composite

$$H^*(X) \otimes H^*(X) \to H^*(X \times X) \to H^*(X)$$

gives a ring operation on $H^*(X)$. Note that this doesn't work to give a coalgebra structure on homology in general precisely because the Kunneth formula for chain complexes goes the wrong way. This doesn work over a field since then the Kunneth formula is an isomorphism. See this question for more details.

The relative cohomology SES is also more geometric than relative homology one. Essentially $C^*(X, A; M)$ consists of functions on simplicial complexes that have support in $X \setminus A$, and it also suggests generalizing to sheaves other than the constant sheaf $\underline{\mathbb{Z}}$.

To show that the cup product is graded commutative on cohomology, consider $\tau : C_n(X) \to C_n(X)$ given by mapping $\sigma = [v_0, ...v_n]$ to $(-1)^{n(n+1)/2}[v_n, ...v_0]$. We want to show that this map is chain homotopic to the identity, which will imply the graded commutativity since chain homotopic maps induce the same maps on homology. In other words, given $\sigma : \Delta^n \to X$ a singular n-complex, we want to produce $P(\sigma)$ a singular chain from Δ^{n+1} to X, essentially the idea is to build up the swapping in a step-by-step fashion: $\Delta^n \times I$ can be triangulated by Δ^{n+1}, say one of them is $[v_0, ...v_k, w_k, ...w_n]$ for some $0 \leq k \leq n$, then we apply σ to the w's, and keep the v's, and take the alternating sum over k. All this is really geometry and combinatorics of simplices, in the sense that X plays no role. To be more precise, on the level of chains $C^*(X; R)$, the difference

$$[\varphi, \varphi] = \varphi \cup \psi - (-1)^{k\ell}(\psi \cup \phi) = \delta(Z_{\phi,\psi})$$

naturally. The proof indeed says on the level of chains, the operation is commutative up to a specified homotopy that is part of the data.