Qualifying exam question (Algebra, Representation theory, Category Theory and Homological Algebra)

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As preparation for my upcoming PhD written preliminary exam, the following are some sample questions in Algebra, Representation theory, Category Theory and Homological Algebra from various sources (thanks to this mathstackexchange post):

1. Let k_0 be the field $\mathbb{Z}/p\mathbb{Z}$ and k algebraically closed field containing k_0. Fix $a \in k$, $P \subset k[X]$ the polynomial $P(X) = X^{p^2} + aX^p + X$. Let K be the field $k(y)$ of rational functions in an indeterminate y.

a. Prove $V := \{x \in k : P(x) = 0\}$ is a 2-dimensional vector space over k_0. (Checking vector space condition is easy; Dimension being 2 follows from $|V| = p^2$.)

b. Let F be the extension of K obtained by adjoining a root x of $P(x) = y$. Show that F/K is a Galois extension with Galois group isomorphic to V. (To check F is Galois over K, note that $P(x+v) = P(x) = y$ for $v \in V$, so F contains all the roots of $P(x) = y$; A bijection from $(V, +)$ to $Gal(F/K)$ is given by $v \mapsto g_v$ where $g_v(x) = x + v$. The picture is that F is the function field of an abelian Galois cover of $\mathbb{P}^1$ of degree p^2.)

c. How many fields E are there such that $K \subset E \subset F$ other than K and F? (By Galois correspondence the intermediate fields correspond to subgroups $H \subset V$ of order p, so there are $(p^1 - 1)/(p-1) = p + 1$ many.)

2. Let G be a finite group, H a normal subgroup of G, P be a Sylow p-subgroup of H.

a. (Frattini's argument) Show that $G = HN_G(P)$ where $N_G(P)$ is the normalizer subgroup of P in G. (Given $g \in G$, since gPg^{-1} is Sylow p-subgroup of H, there exists $h \in H$ such that $hPh^{-1} = gPg^{-1}$.)

b. Show that if every maximal proper subgroup of G is normal in G, then so is any Sylow p-subgroup of G. (If P is not normal, then we can let M be a maximal proper subgroup containing $N_G(P)$. Then Frattini's argument implies that $G = MN_G(P) = M$, contradiction.)

3. Let G be a finite group, $\{\chi_i\}_{1\leq i\leq r}$ irreducible characters of G, H subgroup of G, ψ irreducible character of H. Show that the integers $d_1, ...d_r$ for which $\psi^G = \sum_i d_i\chi_i$ satisfy $\sum_i d_i^2 \leq [G : H]$ (By Frobenius reciprocity, $d_i = \langle\psi, \chi_i|_H\rangle_H$; Since ψ is irreducible, this means that $\chi_i|_H = d_i\psi + \alpha$ for some other α. Thus $\psi(1)[G : H] = \psi^G(1) = \sum d_i\chi_i(1) \geq \sum d_i^2\psi(1)$.)

4. Let $A \in M_n(\mathbb{C})$ be a matrix. Show that $\det(e^A) = e^{trace(A)}$. (The subset of diagonlizable matrix is dense and by continuity argument we are done.)

5. Consider the fields $L = \mathbb{C}(x)$, $K_1 = \mathbb{C}(x^2)$ and $K_2 = \mathbb{C}((x-1)^2)$. Show that $[L : K_1] = [L : K_2] = 2$, but $[L : K_1 \cap K_2] = \infty$. (For the last claim, if $f \in K_1 \cap K_2$, then $f(x) = f(-x) = f(2-x)$, which implies f is constant.)

6. Prove that if V is a finite-dimensional space over a field k, and $(-,-) : V \times V \to k$ is a nondegenerate bilinear pairing such that $(x, x) = 0$ for all $x \in V$, then $\dim_k(V)$ is even. (Find a subspace of codimension 2 such that the restriction of the alternating bilinear form to it is nondegenerate and finish by induction.)

7. Find the number of elements of order 7 in a simple group of order 168. (The number of 7-Sylow subgroups is congruent to 1 mod 7 and divides 24 so it is 8. Each 7-Sylow contains six elements of order 7.)

8. Use the solvability of groups of order 12 to prove that groups of order $588 = 22 \times 3 \times 7 \times 2$ are solvable. (The 7-Sylow subgroup is normal and its quotient has order 12.)

9. If X and Y are objects of a category C, explain succinctly (but precisely) what is meant by the product of X and Y. (The important thing is that the existence of projection maps with universal property.)

10. Let $\mathcal{C}$ be the category with objects being natural numbers and morphisms $Mor(m, n)$ being set of $m \times n$ matrices and composition given by matrix multiplications. Does product exists in this category? (Yes, it is the sum of two natural numbers. Note that this cateogory is equivalent to that of finite dimensional vector spaces.)

11.

a. What is the sign of the permutation given by $x \mapsto gx$ in a group G and g is an element of order l and n is the order of G? (The answer is $((-1)^{l+1})^{n/l}$. Simply note that each orbit has size l and there are n/l many orbits.)

b. Suppose that the 2-Sylow subgroups of G are cyclic and that G has even order. Prove that G has a subgroup of index 2. (There is a sign homomorphism from G to $\{\pm 1\}$ given by the $G \to Perm(G) \to \{\pm 1\}$. The hypothesis together with part (a) shows that the image is not identically 1.)

12. Establish the irreducibility over $\mathbb{Q}$ of each of the following polynomials:

a. $x^{13} + 27x^2 - 120x + 69$ (Eisenstein polynomial)

b. $x^3 + 3x^2 + 9$ (mod 2 is irreducible)

c. $x^3 + x^2 + 2$ (has no roots in $\mathbb{Z}$)

13. Suppose that A is an integral domain (i.e., a commutative entire ring). Suppose that I and J are non-zero ideals of A for which the product IJ is a principal ideal. Show that the ideals I and J are finitely generated. (Suppose $IJ = (a)$ with $a = \sum x_i y_i$. Show that I is generated by x_i.)

14. Show that the covariant functor from Set to Set taking A to $\mathcal{P}(A)$ is nto representable. (It doesn't preserve limits.)

Note: If we consider it as a contravariant functor it is representable by $\{0, 1\}$.

15. Let A be a normal subgroup of order p of a finite p-group G. Prove that A is contained in the center of G. (Consider G acting by conjugation on A and compare orders.)

16. Let F be a finite field, and set $q = |F|$. For each $d \geq 1$, let f_d be the product of monoic irreducible polynomial of degree d over F. Then we have $X^{q^n} - X = \prod_{d|n} f_d$. (Note that $X^{q^n} - X$ is not divisible by square since its derivative is -1; The rest is using theory of finite fields.)

17. Let K/k be a finite Galois extension. Set $G = Gal(K/k)$ and let H be a subgroup of G. Express the group of field automorphisms $\mathrm{Aut}_k(K^H)$ as a quotient of a subgroup of G. (The answer is $N(H)/H$. Any automorphism of K^H can be extended to K, say givin by $g \in Gal(K/k)$, which takes K^H to gK^H. By Galois correspondence the latter corresponds to gHg^{-1}. Thus g is an automorphism of K^H iff g lies in the normalizer of H, and g belong to H iff g acts as identity on K^H.)

18. Let p be a prime number different from 2, and let ζ be a complex p-th root of 1. Set $\alpha = \zeta + \zeta^{-1}$. Show that $\mathcal{Q}(\alpha)$ is a Galois extension of $\mathcal{Q}$ and determine the degree $[\mathcal{Q}(\alpha) : \mathcal{Q}]$. When $p = 7$, calculate $Irr(\alpha, \mathcal{Q}, X)$. (Calculate the number of conjugates for the degree and rewrite the minimal polynomial of ζ to get that of $\zeta + \zeta^{-1}$.)

19. Give a counterexample to show that the image of a functor $F : \mathcal{C} \to \mathcal{D}$ need not be a subcategory. (The problem is that morphisms in the image may not compose.)

20. Showing taking the center of a group cannot be made into a functor. (Construct $G \to H \to K$ such that the composite is an iso and $Z(H) = 1$.)

21. Show that if $|G| = 0$ in k, then the number of isomorphism classes of irreducible representations of G over k is strictly less than the number of conjugacy classes in G. (It suffices to find a class function that is not a linear combination of characters. Note that the linear extension of any irreducible character vanish on the element $P := \sum_{g \in G} g$ in the group algebra since $P^2 = 0$.)

22. Let $H \subseteq G$ be a subgroup. Determine when the character $\chi := Ind_H^G(1) - 1$ is irreducible. (Use Frobenius reciprocity, the answer is iff H has index at most 2.)

23. Let E/F be a field extension. Show that two matrices are similar over E iff they are similar over F. (Use structure theorem of finitely generated modules over PID.)

24. Prove that submodules of free modules over PIDs are free. (For finite rank modules, we can induct on the rank; More generally, we can do transfinte induction, for details see this answer.)

25. Show that there exists a chain homotopy equivalence between a chain complex $X_\bullet$ and $H_n(X_\bullet)$ (with zero differential) iff $X_\bullet$ is split. (Trial and error works, but the more intuitive way is to observe the spliting breaks $X_n = Im(X_{n+1}) \oplus H_n \oplus Im(X_n)$.)

26. Show that equivalence between abelian categories is automatically additive (i.e. preserve biproducts) and exact. (Any equivalence between categories can be upgraded to an adjoint equivalence. Then use the fact that left (resp. right) adjoint is right (resp. left) exact (since it preserves any small colimits (resp. limits)).)

Note: The point is that a category being abelian is a property rather than an additional structure, see this and this for more details.