Intersection theory on moduli space of curves

J'ignore • 11 Jul 2025

What is tautological ring

First defined by Mumford, $R^*(\overline{M_{g,n}})$

What is $\overline{M_{g,n}}$, moduli space of curves of (arithmetic genus g) and n marked (smooth) points, all singularities simple nodes; stability condition: every irreducible component of genus zero must have ≥ 3 distinguished points $2g-2+n$ should be ≥ 0 stability condition gurantees no automorphisms, it is a DM stack smooth and proper of dim 3g-3+n

Intersection theory means studying Chow rings $A^*(\overline{M_{g,n}})$, there is a cycle map from this ring to the cohomology $H^*(\overline{M_{g,n}})$, tautological ring is a subring of the Chow ring, classes naturally arise in geometry. If we restrict this to tautological ring, is it an isomorphism to its image.

Maps between $\overline{M_{g,n}}$: 1. forget a marked point (need to restablize if necessary by contracting unstable component) 2. glueing map

Tautological ring is simultaneously defined as the smallest subring of the Chow ring closed under pushforward under the above maps starting with fundamantal class in each space

Essentially tautological divisors are boundary divisors of specific shapes.

But we can also get chern class?

drawing dual graph of stable curves convey same info

There is a permutation representation of S_n on tautological

Something called Grothendieck-Teichmuller group

profinite version

Goal of Grothendieck: understand $G_{\mathbb{Q}}$

uncountable group, what are its elements, action on $\pi_1^{et}(X_{\overline{\mathbb{Q}}})$ for certain schemes $X/\mathbb{Q}$.

$\pi_1^{et}(X) \cong \pi_1\widehat{(X(\mathbb{C}))}$

LHS is acted on by $G_{\mathbb{Q}}$, so is π_1;

E.g. $X = \mathbb{A}^1 \setminus \{0\} = G_m$, then $\pi_1^{et}(X) \cong \widehat{\mathbb{Z}}$

the action $\chi : G_{\mathbb{Q}} \to Aut(\widehat{\mathbb{Z}}) \cong \widehat{Z}^{\times}$ is the cyclotomic character

Remove one more point: $X = \mathbb{A}^1 \setminus \{0, 1\}$, now $\pi_1(X_{\mathbb{Q}}) \cong \widehat{F_2}$. Belyi prove that this is a faithful representation.

Idea: constrain image by way of various maps behave between moduli space

Drinfeld's subgroup $\widehat{GT} \subset Aut(\widehat{F_2})$ subgroup of the form $\varphi : \widehat{F_2} \to \widehat{F_2}$: Let x, y be the two generators of F_2, $\varphi(x) = x^{\lambda}, \varphi(y) = f^{-1} y^{\lambda} f$, $\lambda \in 1 + 2\widehat{\mathbb{Z}}$, $f \in [\widehat{F_2}, \widehat{F_2}]$ such that 1. $f(y, x) = f(x, y)^{-1}$ 2. $f(z, x) z^m f(y, z) y^m f(x, y) x^m = 1$ if $m = \frac{1-\lambda}{2}$. 3. $f(x_{12}, x_{23} x_{24}) f(x_{13} x_{23}, x_{34}) = f(x_{23}, x_{34}) f(x_{12} x_{13}, x_{24} x_{34}) f(x_{12}, x_{23})$ where x_{ij} is generator of PB_4. Theorem (Drinfeld-Ihara): Image is contained in $\widehat{GT}$ conjecture is it is exactly that

Connection to topology: configuration space $PConf_n = \{(x_1, ...x_n) : x_i \neq x_j\}$ (ordered configuration space), $Conf_n$ unordered analog. $\pi_1(Conf_n(\mathbb{C})) = B_n$ braid group, $\pi_1(PConf_n(\mathbb{C})) = PB_n$. $PConf_3 \to PConf_2$ is a fibration with fiber over $(0,1) equal to $\mathbb{A}^1 \setminus \{0, 1\}$, $PConf_2$ is isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1 \setminus \{0\}$, fibration split so $\pi_1(PB_3) = \pi_1 PConf_3 \cong \mathbb{Z} \times F_2$

Braid group appear as automorphisms of objects in braided monoidal category

Drinfeld's definition: mess with all structures of bracat

Notice: $PConf_n(\mathbb{C}) \cong E_2(n)$ little disks operad

Theorem (Horel, based on Drinfeld, Bar-Natan) $\widehat{GT} \cong hAut(\widehat{E_2})$

The free algerba $Lie[V]$ is graded (i.e. bracket preserves degree); There should exist an adjunction from category of lie algebras to that of associative algebras making the diagram of free functors commute. Indeed, this is the functor of forming universal envelopping algebra.

A k-coalgebra C is a vector space equipped with a map $\Delta : C \to C \otimes C$ and a counit $C \to k$ such that it is coassociative and satisifies counitality ($C \to C \otimes C \to C \otimes k \cong C$ is identity and another one for right counitality) Bialgebra: Δ is ring hom

Monoidal cat - $\alpha_{x,y,z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $X \otimes 1 \cong 1 \otimes X \cong X$ braiding - $\sigma : X \otimes Y \to Y \otimes X$, symmetric if $\sigma^2 = id$ Example: product of two sets, tensor product of vector spaces, graded vector space with braiding given by Koszul sign convention

An algebra in a monoidal category is a monoidal object A (bilinearity is distributive law)

If $\mathcal{C}$ is braided and A, B are algebras, then so is $A \otimes B$.

A bialgebra in a braided category is a algebra and coalgebra s.t. comultiplication is a mp of algebras. It is moreover a Hopf algebra if there is an antipode (like inverse) $S : A \to A$ such that $A \to A \otimes A \xrightarrow{S \otimes 1} A \otimes A \to A$ is equal to the composite $A \to 1 \to A$.

Example: group algebra, universal envelopping algebra of Lie algebra (Milnor-Moore: There is an equivalence of category between the category of Lie algebras to that if primitively generated Hopf algebras if k is characterstic zero, one direction is given by taking universal envelopping algebra, the other is given by taking primitive elements)

Let $G(A)$ be the set of group-like elements, then there is a map of algebras $k[G(A)] \to A$.

(Cartier-Kostant-Milnor-Moore) If A is Hopf algebra over algebraically closed field, and characteristic 0 and co-commutative, then $A \cong \mathcal{U}(P(A)) \rtimes k[G(A)]$ (does use algebraically closed)

impplies if A is generated by group like elements, then A is isomorphic to $k[G(A)]$

$\mathcal{U}(Lie[V]) \cong T(V)$ natural isomorphism (same universal property)

Completed tensor product: $\widehat{V \otimes W} := \widehat{V} \otimes \widehat{W}$, but completion of vector spaces are not unique, https://en.wikipedia.org/wiki/Complete_topological_vector_space, though there is a unique Hausdorff completion. We want to make the category of TVS with completed tensor product to be a monoidal category

If A is filtered by $\mathfrak{i}$, we can define the completion $\widehat{A}$ to be the inverse limit of $A/\mathfrak{m}_i$ and the topology is given by the local base $\mathfrak{i}$; $\widehat{A}$ can be filtered

$\widehat{A} \widehat{\otimes} \widehat{B} := \varprojlim A/\mathfrak{m}_i \otimes B/\mathfrak{n}_j$

If V is graded vector space $V \cong \oplus V_n$, then $\widehat{V} \cong \prod V_n$

If L is graded Lie algebra, then $\widehat{L}$ is completed Lie algbra (need to use that L is positively graded), but it is not graded vector space.

Filter: $\mathfrak{m}_k \subseteq \mathcal{U}(\widehat{L})$ be the ideal generated by $x_1 ... x_l$ such that $x_l \in \widehat{L}_{\geq k_i} := \prod_{n \geq k_i} L_n$ such that $\sum k_i \geq k$. Let $\widehat{\mathcal{U}}(\widehat{L}) := \varprojlim \mathcal{U}(\widehat{L})/\mathfrak{m}_k$. $\widehat{\mathcal{U}}(\widehat{L})$ then becomes a completed Hopf algebra, and it is isomorphic to $\widehat{\mathcal{U}(L)}$

Let $\mathfrak{g}$ be a complete positive graded Lie algebra, over a field of characteristic 0. Define $G := \exp(\mathfrak{g})$, we define multiplication $\exp(x) \cdot \exp(y) := \log(e^x e^y)$. A priori this only makes sense in $\widehat{\mathcal{U}(\mathfrak{g})}$, but the content of BCH is that this has an alternate expression
$bch(x, y) = x + y + 1/2[x, y] + 1/12[x, [x, y]] - 1/12[y, [y, x]] + ...$

Example: If $\mathfrak{g}$ is the completion of the free Lie algebra $Lie(V)$, there is an isomorphism of groups from $\exp(\mathfrak{g})$ to $G(\widehat{\mathcal{U}(\mathfrak{g})}) \cong \widehat{T(V)}$ the group like element of the completed Hopf algebra (It is important to complete it, since the only group-like elements in the usual envelopping algebra is 1 from PBW theorem; conversely, the only primitive element in a group Hopf algebra is 0, as can be checked by direct computation.)

Sketch: $\widehat{\mathfrak{g}} = P(\widehat{\mathcal{U}}(\widehat{\mathfrak{g}}))$, and $x \in \widehat{\mathcal{U}}(\widehat{\mathfrak{g}})$ is primitive iff e^x is group-like $(\Delta(e^x) = e^{\Delta(x)} = e^{x\otimes 1}e^{1\otimes x} = e^x \otimes e^x$.

Let $\mathcal{C}$ be a symmetric monoidal cateogory. Let X be an object and let $End_X(n) = Hom_{\mathcal{D}}(X^{\otimes n}, X)$. There is an action of S^n on $End_X(n)$ (transposition acts by the braiding, which is ok since it is symmetric; we also need to check $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $i > j + 1$ and $(\sigma_i\sigma_{i+1})^3 = 1$ which follows from the braiding axiom)

There is an insertion operation: for every $n, m \in \mathbb{N}$ and $i \in \{1, ..., m\}$, $\circ_i : End_X(n) \times End(m) \to End(n + m - 1)$

Sometimes $End_(n)$ has more structure than just a set, can ask that $\mathcal{C}$ is enriched over another monoidal category $\mathcal{D}$, i.e. $Hom_{\mathcal{C}}(A, B)$ is an object in $\mathcal{D}$, and composition is a morphism in $\mathcal{D}$, and we also want to ask $\mathcal{C}$ is cotensored over $\mathcal{D}$, i.e. we can take tensor product of an object in $\mathcal{C}$ and $\mathcal{D}$ and also $Hom(A, B)$ as objects in $\mathcal{C}$ satisfying various axiom. For example, if $\mathcal{D}$ is a subcategory of $\mathcal{C}$, e.g. the category of vector space is cotensored over that of sets by defining $X \otimes V = \oplus_{x\in X} V$.

Let $\mathbb{S}$ be the groupoid of finite sets. The skeletons are $\mathbb{Z}_{\geq 0}$ and morphisms are from an object to itself and equal to S_n. A symmetric sequence is a $\mathbb{S}$-module in a category $\mathcal{C}$ is a functor $\mathcal{O} : \mathbb{S} \to C$, determined up to natural equivalence by restriction to sk, so it is nothing but a colletion of objects $\mathcal{O}(n)$ with an action of S_n on $\mathcal{O}(n)$. An operad $\mathcal{O}$ in C is a symmetric sequence with maps, for any I, J finite sets, $i \in I$, we have an morphism $\circ_i^{I,J} : \mathcal{O}(I)\otimes)\mathcal{C}\mathcal{O}(J) \to \mathcal{O}(I \setminus \{i\} \sqcup J)$, and it should be natural in $(i, I), J$ (equivariant condition), which are associative: 1. Plugging in two different slots of I commute 2. Plugging in I then in J is the same as first plugging I into J then plugging into I. There is also a unital axiom.

Example: if $\mathcal{D}$ is enriched, cotensored over $\rfloor$, then $\{End_X(n)\}$ is an operad. If A is a finite set, then $End_X(A) = Hom(X^{\otimes A}, X)$ where $X^{\otimes A}$ is the tensor product of X $|A|$ times, each copy labelled by elements of A.

Let nature do its work and forget

An algebra structure on an object X over an operad $\mathcal{O}$ is a morphism $\mathcal{O} \to End_X$ between operads. If there is a hom-tensor adjunction then this is saying $\mathcal{O}(n) \otimes_{S_n} X^{\otimes n} \to X$.

Examples: $Assoc(n) = S_n$, with right action of S_n on itself; $S_n \times S_m \to S_{n+m-1}$, $i \in \{1, ..., n\}$ is given by block insertion

Why is it called $Assoc$? Because an object A in a monoidal category is an associative algebra iff it has the structure of $Assoc$-algebra, i.e. $Hom_{op}(Assoc, End_A) \cong$ monoidal structure on A ($id \otimes A^{\otimes 2} \to A \to A$ gives the multiplication. The other direction is permuting the factor); Also note that the associativity axioms in operads means that insertion of operations are associative, it doesn't mean that the operation is associative in general (but it doesn in this case).

If $\mathcal{O}$ is an operad in $(C, \otimes, \not\Vdash)$, and $F : C \to C'$ is a (lax) monoidal functor, then $F(\mathcal{O})$ is an operad in C'. The associative operad in k-modules is $F(Assoc)$, explicitly, the n-th term is $k[S_n]$

Operad of parenthesized mutations $P_a : \mathbb{S} \to Set$, e.g. $I = \{1, 2, 3, 4\}$, (12)(34), ((4(23))1)$ are elements of $P_a(I)$, equivalently $P_a(I)$ is the set of binary rooted trees labelled by I.

Little disk operads: A TD-map is a function $f : D^n \to D^n$ from the open unit disk to itself of the form $ax + b$ where $a > 0$ and $b \in \mathbb{R}^n$. $E_n(k) := \{(f_1, ...f_k) : f_i TD, im(f_i) \text{ pairwise disjoint}\}$, this can be topologized as a subspace of $(\mathbb{R}_{>0} \times \mathbb{R}^n)^k$

Commutative operad: $Com(n) = \{*\} = (S_n)_{S_n}$. Structure maps are forced by definition. There is a map of operad $Assoc \to Com$, so if A is a Com-algebra, it is also a Assoc-algebra. THere exists map of operads $\mathcal{P} \to Assoc$ by forgetting the parentheses. What is a $\mathcal{P}$-algebra, it is just binary operations on X.

In $Vect_k$, recall $Assoc(n) = k[S_n]$, then the Lie operad is the suboperad generated by $[-, -] := id - (12) \in Assoc(2)$. Jacobi idenitty is forced by the suboperad. We will see that Lie is freely generated by $Lie(2) = k[-, -]$ with the sign representation of S_2 subject to the Jacobi identity.

In this language universal envelopping algebras are adjoint to forgetful functor

$\mathcal{P}$ is free operad generated by $(12) \in \mathcal{P}(2)$.

Free operads: Left adjoint to the forgetful functor from operads to symmetric sequences

Graph:= collection of half-edges (connected to one vertex), a vertex is an equivalence class of half edges, an involution on the half edges, fixed points are legs, internal edges are pairs of half edges swapped under involution

We can produce a 1D CW complex homeomorphic to a graph. A tree is a graph whose top space is simply connected, and rooted tree is a tree with a leg picked as a root, the other legs are leaves. Since the graph is a 1-dimensional simply connected there exists unique path from any vertex from leaf to the root, and we direct the edge according to it. for any vertex v, we have the incoming half edges $I(T)$ and the outgoing half edges.

Digression: how to start from $\mathcal{O}(n)$ each one with an action of S_n to get a functor $\mathcal{O} : \mathbb{S} \to \mathcal{C}$: $\mathcal{O}(I) = (\sqcup_{\rho:[n]\to I}\mathcal{O}(n))_{S_n}$, and this natural in I (by postcomposing with $I \to J$)

Explicit description of free operad: Let $\mathcal{E}$ be a symmetric sequence: Define $F(\mathcal{E})(I)$ for some finite set $I = \{i_1, ...i_n\}$. The idea is direct summing over all trees and all labels on leaves by I of a certain object in $\mathcal{C}$. The object is

$$T\langle\mathcal{E}\rangle = \otimes_{v\in V(T)}\mathcal{E}(I_n(V))$$

and we can define $F(\mathcal{E})(I) = colim_T T\langle\mathcal{E}\rangle$

Ideals of operads: collection of subobject $I(n) \subset \mathcal{O}(n)$ closed under insertion from left. A presentation of an operad is an isomorphism $Free(P)/I \to \mathcal{O}$

Examples: In Set, Pa is the free operad on Q where Q is a symmetric sequence with S_2 at degree 2 and $\emptyset$ otherwise. In vec, $Assoc$ is $Pa/(\mu \circ_1 \mu - \mu \circ_2 \mu)$ where μ is the nonidentity element of S_2 (see Ginzburg-Kapranov Koszul duality for operads for reference). Similarly for the Lie operad (proof?)

Recall $PB_n(X)$ is $\pi_1(Pconf_n(X), x)$ and similarly for $B_n(X)$, it is independent of x if X is a connected manifold with $\dim > 1$

The intuition is that we can think of a loop in the (paranthesized) configuration space of $\mathbb{C}$ as braids in three dimensions by thinking of time as the vertical axis.

Yetter-Drinfeld modules: G group, yD_G^G be the category with object right $k[G]$-modules which decompose $V = \oplus_{g\in G}V_g$ such that $V_g \cdot h = V_{g^h}$ where $g^h = h^{-1}gh$ and morphisms are linear maps preserving the action and the grading

Monoidal structure: $(V \otimes W)_g = \oplus_{g_1g_2=g}V_{g_1} \otimes W_{g_2}$ with action via diagonal action.

Braiding: $\sigma : V \otimes W \to W \otimes V$ is going to send $V_{g_1} \otimes W_{g_2}$ to $W_{g_2} \otimes V_{g_1^{g_2}}$ (similar to semidirect product) where $v \otimes w \mapsto w \otimes (v \cdot g_2)$

Monoidal functor (lax): $F : C \to D$ is monoidal if there exists natural transformation $F(A) \otimes F(B) \to F(A \otimes B)$ (colax if it is the other direction)

If A is an algebra in C and F is lax monidal then $F(A)$ is an algebra in D (coalgebra if it is colax monoidal)

operad - trees + graph complexes (Ginzburg - Kapranov) Koszul duality for operad

- May's Geometry of iterated loop space (older definition)
- smirnov - operad as monoid in certain category (symmetric sequence)