

Intersection theory on moduli space of curves

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What is tautological ring

First defined by Mumford, $R^*(\overline{M}_{g,n})$

What is $\overline{M}_{g,n}$, moduli space of curves of (arithmetic genus g) and n marked (smooth) points, all singularities simple nodes; stability condition: every irreducible component of genus zero must have ≥ 3 distinguished points $2g - 2 + n$ should be ≥ 0 stability condition guarantees no automorphisms, it is a DM stack smooth and proper of $\dim 3g-3+n$

Intersection theory means studying Chow rings $A^*(\overline{M}_{g,n})$, there is a cycle map from this ring to the cohomology $H^*(\overline{M}_{g,n})$, tautological ring is a subring of the Chow ring, classes naturally arise in geometry. If we restrict this to tautological ring, is it an isomorphism to its image.

Maps between $\overline{M}_{g,n}$: 1. forget a marked point (need to restablize if necessary by contracting unstable component) 2. glueing map

Tautological ring is simultaneously defined as the smallest subring of the Chow ring closed under pushforward under the above maps starting with fundamental class in each space

Essentially tautological divisors are boundary divisors of specific shapes.

But we can also get chern class?

drawing dual graph of stable curves convey same info

There is a permutation representation of S_n on tautological

Something called Grothendieck-Teichmüller group

profinite version

Goal of Grothendieck: understand $G_{\mathbb{Q}}$

uncountable group, what are its elements, action on $\pi_1^{et}(X_{\overline{\mathbb{Q}}})$ for certain schemes X/\mathbb{Q} .

$$\pi_1^{et}(X) \cong \pi_1(\widehat{X(\mathbb{C})})$$

LHS is acted on by $G_{\mathbb{Q}}$, so is π_1 ;

E.g. $X = \mathbb{A}^1 \setminus \{0\} = G_m$, then $\pi_1^{et}(X) \cong \widehat{\mathbb{Z}}$

the action $\chi : G_{\mathbb{Q}} \rightarrow \text{Aut}(\widehat{\mathbb{Z}}) \cong \widehat{\mathbb{Z}}^\times$ is the cyclotomic character

Remove one more point: $X = \mathbb{A}^1 \setminus \{0, 1\}$, now $\pi_1(X_{\mathbb{Q}}) \cong \widehat{F}_2$. Belyi prove that this is a faithful representation.

Idea: constrain image by way of various maps behave between moduli space

Drinfeld's subgroup $\widehat{GT} \subset \text{Aut}(\widehat{F}_2)$ subgroup of the form $\varphi : \widehat{F}_2 \rightarrow \widehat{F}_2$: Let x, y be the two generators of F_2 , $\varphi(x) = x^\lambda$, $\varphi(y) = f^{-1}y^\lambda f$, $\lambda \in 1 + 2\widehat{\mathbb{Z}}$,

$f \in [\widehat{F}_2, \widehat{F}_2]$ such that 1. $f(y, x) = f(x, y)^{-1}$ 2.

$f(z, x)z^m f(y, z)y^m f(x, y)x^m = 1$ if $m = \frac{1-\lambda}{2}$. 3.

$f(x_{12}, x_{23}x_{24})f(x_{13}x_{23}, x_{34}) = f(x_{23}, x_{34})f(x_{12}x_{13}, x_{24}x_{34})f(x_{12}, x_{23})$ where x_{ij} is generator of PB_4 . Theorem (Drinfeld-Ihara): Image is contained in \widehat{GT} conjecture is it is exactly that

Connection to topology: configuration space $PConf_n = \{(x_1, \dots, x_n) : x_i \neq x_j\}$ (ordered configuration space), $Conf_n$ unordered analog. $\pi_1(Conf_n(\mathbb{C})) = B_n$ braid group, $\pi_1(PConf_n(\mathbb{C})) = PB_n$. $PConf_3 \rightarrow PConf_2$ is a fibration with fiber over $(0,1)$ equal to $\mathbb{A}^1 \setminus \{0, 1\}$, $PConf_2$ is isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1 \setminus \{0\}$, fibration split so $\pi_1(PB_3) = \pi_1 PConf_3 \cong \mathbb{Z} \times F_2$

Braid group appear as automorphisms of objects in braided monoidal category

Drinfeld's definition: mess with all structures of braided

Notice: $PConf_n(\mathbb{C}) \cong E_2(n)$ little disks operad

Theorem (Horel, based on Drinfeld, Bar-Natan) $\widehat{GT} \cong h\text{Aut}(\widehat{E}_2)$

The free algebra $\text{Lie}[V]$ is graded (i.e. bracket preserves degree); There should exist an adjunction from category of lie algebras to that of associative algebras making the diagram of free functors commute. Indeed, this is the functor of forming universal enveloping algebra.

A k -coalgebra C is a vector space equipped with a map $\Delta : C \rightarrow C \otimes C$ and a counit $C \rightarrow k$ such that it is coassociative and satisfies counitality

($C \rightarrow C \otimes C \rightarrow C \otimes k \cong C$ is identity and another one for right counitality)

Bialgebra: Δ is ring hom

Monoidal cat - $\alpha_{x,y,z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$, $X \otimes 1 \cong 1 \otimes X \cong X$

braiding - $\sigma : X \otimes Y \rightarrow Y \otimes X$, symmetric if $\sigma^2 = id$ Example: product of two sets, tensor product of vector spaces, graded vector space with braiding given by Koszul sign convention

An algebra in a monoidal category is a monoidal object A (bilinearity is distributive law)

If \mathcal{C} is braided and A, B are algebras, then so is $A \otimes B$.

A bialgebra in a braided category is a algebra and coalgebra s.t. comultiplication is a mp of algebras. It is moreover a Hopf algebra if there is an antipode (like inverse) $S : A \rightarrow A$ such that $A \rightarrow A \otimes A \xrightarrow{S \otimes 1} A \otimes A \rightarrow A$ is equal to the composite $A \rightarrow 1 \rightarrow A$.

Example: group algebra, universal enveloping algebra of Lie algebra (Milnor-Moore: There is an equivalence of category between the category of Lie algebras to that of primitively generated Hopf algebras if k is characteristic zero, one direction is given by taking universal enveloping algebra, the other is given by taking primitive elements)

Let $G(A)$ be the set of group-like elements, then there is a map of algebras $k[G(A)] \rightarrow A$.

(Cartier-Kostant-Milnor-Moore) If A is Hopf algebra over algebraically closed field, and characteristic 0 and co-commutative, then $A \cong \mathcal{U}(P(A)) \rtimes k[G(A)]$ (does use algebraically closed)

implies if A is generated by group like elements, then A is isomorphic to $k[G(A)]$

$\mathcal{U}(\text{Lie}[V]) \cong T(V)$ natural isomorphism (same universal property)

Completed tensor product: $\widehat{V \otimes W} := \widehat{V} \otimes \widehat{W}$, but completion of vector spaces are not unique, <https://en.wikipedia.org/wiki/>

Complete_topological_vector_space, though there is a unique Hausdorff completion. We want to make the category of TVS with completed tensor product to be a monoidal category

If A is filtered by \mathfrak{i} , we can define the completion \widehat{A} to be the inverse limit of A/\mathfrak{m}_i and the topology is given by the local base \mathfrak{i} ; \widehat{A} can be filtered

$$\widehat{A} \widehat{\otimes} \widehat{B} := \varprojlim A/\mathfrak{m}_i \otimes B/\mathfrak{n}_j$$

If V is graded vector space $V \cong \oplus V_n$, then $\widehat{V} \cong \prod V_n$

If L is graded Lie algebra, then \widehat{L} is completed Lie algebra (need to use that L is positively graded), but it is not graded vector space.

Filter: $\mathfrak{m}_k \subseteq \mathcal{U}(\widehat{L})$ be the ideal generated by $x_1 \dots x_l$ such that $x_l \in \widehat{L}_{\geq k_l} := \prod_{n \geq k_l} L_n$ such that $\sum k_i \geq k$. Let $\widehat{\mathcal{U}}(\widehat{L}) := \varprojlim \mathcal{U}(\widehat{L})/\mathfrak{m}_k$. $\widehat{\mathcal{U}}(\widehat{L})$ then becomes a completed Hopf algebra, and it is isomorphic to $\widehat{\mathcal{U}(L)}$

Let \mathfrak{g} be a complete positive graded Lie algebra, over a field of characteristic 0. Define $G := \exp(\mathfrak{g})$, we define multiplication $\exp(x) \cdot \exp(y) := \log(e^x e^y)$. A priori this only makes sense in $\widehat{\mathcal{U}(\mathfrak{g})}$, but the content of BCH is that this has an alternate expression

$$bch(x, y) = x + y + 1/2[x, y] + 1/12[x, [x, y]] - 1/12[y, [y, x]] + \dots$$

Example: If \mathfrak{g} is the completion of the free Lie algebra $Lie(V)$, there is an isomorphism of groups from $\exp(\mathfrak{g})$ to $G(\widehat{\mathcal{U}(\mathfrak{g})}) \cong \widehat{T(V)}$ the group like element of the completed Hopf algebra.

Sketch: $\widehat{\mathfrak{g}} = P(\widehat{\mathcal{U}(\mathfrak{g})})$, and $x \in \widehat{\mathcal{U}(\mathfrak{g})}$ is primitive iff e^x is group-like
 $(\Delta(e^x) = e^{\Delta(x)} = e^{x \otimes 1} e^{1 \otimes x} = e^x \otimes e^x$.

completion is complete?