Proof of Mordell-Lang for function fields in characteristic zero using differential algebra

J'ignore • 4 Jul 2025

The Mordell-Lang conjecture is a generalization of the well-known Faltings's theorem on finiteness of rational points on curves of genus at least 2. The statement is roughlt stated as follows:

> Let K be a number field and E an abelian variety over K. The intersection of an algebraic subvariety X with a subgroup of finite rank Γ is contained in a finite union of cosets (of abelian subvarieties of E) contained in X.

Thus the points of $X \cap \Gamma$ are captured by finitely many translates of abelian subvarieties. It implies Faltings's theorem using Jacobian and the Abel-Jacobi map. The purpose of this post is try to explain Anand Pillay's proof Mordell-Lang for function fields in characteristic zero. The primary reference is his 2004 paper Mordell–Lang conjecture for function fields in characteristic zero, revisited.

The formal statement is as follows:

> Let K/k be algebraically closed fields of characteristic zero and A an abelian variety over K. Let X be an irreducible subvariety of A over K and Γ a finite rank subgroup of $A(K)$ (i.e. $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ is a finite-dimensional $\mathbb{Q}$-vector space). Suppose $X \cap \Gamma$ is Zariski dense in X and X has trivial stablizer in A, then up to a translate X is contained in an abelian subvariety of A that descends to an abelian variety defined over k.

This is firstly proved by Buium in 1992. The positive characteristic case is first done by Hrushovski using model-theoretic method (Zilber's trichotomy for Zariski geometry). Note that the conclusion here is stronger than the one in Hrushovski's paper, since we assume in addition that X has trivial stablizer.

A result from complex geometry using deformation-theoretic method

To motivate Pillay's proof we mention the following amazing result due to Ueno:

> If A is a complex torus and X an analytic subvariety of A with trivial stabilizer, then X is an algebraic variety.

This feels like a Chow's theorem for complex tori: Even though complex tori are not algebraic in general, their 'generic' analytic subvarieties are. The proof of these results illustrate the power of taking derivative in diophantine questions over function fields. (E.g. Fermat's last theorem for function fields is a triviality simply by differentiating the equation $f(t)^n + g(t)^n = h(t)^n$.) In fact, I believe the Gauss-Manin connection (a way to differentiate cohomology classes) plays a crucial role in all approaches to Mordell-type conjectures, in one way or another. As an example, a recent approach to Faltings's theorem by Lawrence-Venkatesh uses the p-adic period mapping, whose construction boils down to the Gauss-Manin connection.

Going back to Ueno's result, it is an easy corollary of the following infinitesimal version of Chow's theorem by Campana.

> Let Z be an analytic space, $(X_s)_{s\in S}$ an analytic family of analytic compact cycles in Z parametrized by a compact analytic space S and $X \subseteq S \times Z$ denote the graph of $(X_s)_{s\in S}$. There is a bimeromorphic (birational?) embedding of X into $Gr_r(\mathcal{D}_p(Z))$ for some $p, r \geq 0$ that is compatible with projection to Z. Here $\mathcal{D}_p(Z)$ denotes the sheaf of differential operators of order $\leq p$ and $Gr_r(\mathcal{D}_p(Z))$ is the total space of the Grassmanian sheaf of r-dimensional subspaces of $\mathcal{D}_p(Z)$.

This thoerem implies that a stablizer-free analytic subvariety $X \subset T$ is Moishezon (having enough meromorphic functions), and being a subvariety of the complex torus T it also Kähler, hence projective by Moishezon's theorem.

This suggests deformation-theoretic method (combined with some finiteness assumption such as compactness) can be useful in showing algebracity result, once we are in a setting that allows us to take derivative. One natural setting for doing this is differential algebraic groups.

Preliminaries on differential algebra

We first define what an algebraic D-group is and this requires the language of differential algebra. A good reference is this paper by Pillay. Let K be a field (characteristic zero) with a derivation ∂ and k the field of constant (elements killed by ∂). Note that WLOG, we can extend scalar to make K *differentially closed* (this means that any finite system of differential polynomial equations and inequations over K that has a solution in some extension of K already has a solution over K, analogous to being algebraically closed). An *algebraic D-variety* (defined by Buium) is an irreducible variety X over K equipped with a derivation ∂_X on the structure sheaf of X extending ∂ (so it allows us to differentiate sections in $X(K)). If G is an algebraic group over K and ∂_G is compatible with the multiplication and inversion of G, then G is an *algebraic D-group*.

We define the *∂-twisted tangent bundle* $\tau(X)$ as follows: If X is locally cut out by $P_1(X_1, ...X_n) = 0, ...P_r(X_1, ...X_n) = 0$ over K, then $\tau(X)$ is locally cut out by these equations together with

$$\sum_{i=1}^{n} \left(\frac{\partial P_j}{\partial X_i}\right) v_i + P_j^{\partial} = 0$$

for $j = 1, ..., r$ where $v_1, ...v_n$ are n new variables and P_j^{∂} denotes the polynomial obtained by differentiating the coefficients of P_j with ∂. Note that if X is defined over k, the coefficients of P_j are in k and thus annhilated by ∂, in which case $\tau(X)$ is the tangent bundle of X.

Note that if G is an algebraic D-group, there is a natural algebraic D-group structure on $\tau(G)$ given by sending

$$(x, u) *_{\tau(G)} (y, v) = (x *_G y, d_{(x,y)} *_G (u, v)).$$

We also have a natural algebraic D-group homomorphism $p : \tau(G) \to G$ given by projection.

Pillay actually define *algebraic D-group* in a more explicit way, by replacing the derivation ∂_G with the equivalent datum of a section $s : G \to \tau(G)$ which is also a homomorphism over K; it gives a K-rational splitting of $\tau(G)$ as a semidirect product of G and $\mathrm{Lie}(G)$, the Lie algebra of G thought of as a K-variety. This also amounts to the datum of a K-rational $h : \tau(G) \to \mathrm{Lie}(G)$ that is a left inverse to p and a crossed homomorphism, i.e. we have $h(xy) = h(x) + xh(y)x^{-1}$ for $x, y \in \tau(G)$. We call $X \subset G$ a *D-subvariety* if s maps X to $\tau(X)$.

Arguably the most important definition is the following: giving an algebraic D-variety (X, s), we define $(X, s)^\#$ (or $X^\#$ if s is understood) to be

$$(X, s)^{\#} := \{x \in X(K) : s(x) = \partial(x)\}.$$

If s is the zero section and G is defined over k, this is precisely $G(k)$, which is a much smaller object than $G(K)$. (Pillay says it is a finite-dimensional differential algebraic group, I haven't looked into the definition of it but it presumably has something to do with some model-theoretic dimension?) On the other hand, $G(k)$ is Zariski-dense in $G(K)$ (this reminds me of the fact that the $\overline{\mathbb{Q}}$-points in an algebraic variety $X \subset \mathbb{C}^n$ defined over $\mathbb{Q}$ is Zariski dense, for solution see this mathoverflow post).

Main theorem

Now we can state and prove an analogous algebracity result in this differential algebraic setting:

> Suppose that (G, s) is an algebraic D-group, X is a D-subvariety of (G, s) with trivial stablizer. Assume also that $e \in X$ and X generates G. Then (G, s) comes from k, i.e. there exists an algebraic group G_0 defined over k such that (G, s) is isomorphic to (G_0, s_0) where s_0 is the zero section.

The conclusion is commonly called *(strongly?) isotrivial* in deformation theory literature.

Proof sketch: The p*-jet* $j_p(G)_e$ of G at $e \in G$ is the dual space of $\mathfrak{m}/\mathfrak{m}^{p+1}$ where $\mathfrak{m}$ is the maximal ideal of the local ring of G at e. This is a finite dimensional K-vector space. Similarly, we define $j_p(Y)_e$ for any subvariety of G containing e. The general principle is again

> If Y is a member of an algebraic family of subvarieties, all passing through e,then Y is determined (in this family) by $j_p(Y)_e \subseteq j_p(G)_e$ for sufficiently large p.

In this case this is almost immediate since the family is algebraic to begin with. Since X has trivial stablizer, for $t_1, t_2 \in G$, we have $t_1^{-1}X = t_2^{-1}X$ implies $t_1 = t_2$. If $t \in X$, $t^{-1}X$ contains e. Thus by the general prinicple, for sufficiently large p, the map taking $t \in X \mapsto j_p(t^{-1}X)_e$ gives a birational embedding h of X into $Gr_r(j_p(G)_e)$. The dual space $V = \mathfrak{m}/\mathfrak{m}^{p+1}$ is equipped with a connection D_V over F, which in turn induces one on $j_p(G)_e = V^*$. Since K is differentially closed, we have a fundamental system of solutions of the equation $D_{V^*} = 0$, that is, there exists a tuple $d = (d_1, ..., d_n)$ of elements of V^*

which is simultaneously an K-basis of V^* and a k-basis of the solution space $(V^*)^\partial$. (This certainly reminds me of some constructions in Riemann-Hilbert correspondence and p-adic Hodge theory.)

Now suppose $t \in X^\#$, then $t^{-1}X$ is a D-variety of G, and essentially the same argument as above applies and shows that $W_t = j_p(t^{-1}X)_e$, a ∂-submodule of V^*, admits a finite F-basis that is also a k-basis for the solution space W_t^∂. Thus W_t is a k-rational point of $Gr_r(K^n)$ and we have obtained a birational isomorphism h of X with a subvariety Y of $Gr_r(K^n)$ such that for generic $t \in X^\#$, $h(t)$ is rational over k. Note that, as $X^\#$ is Zariski-dense in X, Y is defined over k and it is not hard to take it from here.

To apply this theorem to Mordell-Lang it remains to produce a finite-dimensional differential algebraic subgroup of $A(F)$ containing Γ. This is done by Buium in the paper mentioned above. More precisely, his construction gives the following:

> There is a connected commutative algebraic D-group (G, s) and a surjective homomorphism (of algebraic groups) $\pi : G \to A$ such that:
>
> 1. $L = ker(\pi)$ is unipotent (and is thus the unipotent radical of G,as A is abelian);
> 2. $\pi|_{G^\#}$ is injective;
> 3. $\Gamma \subset \pi(G^\#)$.

Using this it is not hard to finish the proof. Maybe I will come back and revisit his construction, as it seems rather magical!