When the Discrete Topology Does Not Imply the Discrete Metric

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Metric Space

You may be familiar with the **discrete metric** on an arbitrary set X. The discrete metric makes every subset of X an open set. If we let $\mathcal{T}_X$ denote the topology induced by the discrete metric on X, then

$$\mathcal{T}_X = \{A \mid A \subseteq X\}.$$

Now, is the converse true?

Recall that the discrete metric function e on $X \times X$ is defined as:

$$e(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

So, we can say that every Cauchy sequence in (X, e) is convergent in X.

Suppose we have a metric d on X such that the topology induced by d is

$$\mathcal{T}_X = \{A \mid A \subseteq X\}.$$

Does this imply that d is the discrete metric? **No.**

Counterexample

Define a metric $d \colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ as:

$$d(m, n) = \left| \frac{1}{n} - \frac{1}{m} \right|,$$

where $|\cdot|$ denotes the usual Euclidean metric on $\mathbb{R}$.

- Every subset of $\mathbb{N}$ is open in this metric space. Thus,

$$\mathcal{T}_{\mathbb{N}} = \{A \mid A \subseteq \mathbb{N}\}.$$

- However, this space is **not complete**. Consider the sequence $x_n = n$. It is Cauchy but **not** convergent in $(\mathbb{N}, d)$.

This counterexample shows that even if the topology induced by a metric is discrete topology, the metric itself need not be the discrete metric.

Moreover, if you are familiar with the concept of **equivalent metrics**, observe that $(\mathbb{N}, d)$ is homeomorphic to the metric space $(\mathbb{N}, |\cdot|)$. Hence, this example illustrates that **completeness is not preserved under homeomorphism**.