

# When the Discrete Topology Does Not Imply the Discrete Metric

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## Metric Space

You may be familiar with the **discrete metric** on an arbitrary set  $X$ . The discrete metric makes every subset of  $X$  an open set. If we let  $\mathcal{T}_X$  denote the topology induced by the discrete metric on  $X$ , then

$$\mathcal{T}_X = \{A \mid A \subseteq X\}.$$

Now, is the converse true?

Recall that the discrete metric function  $e$  on  $X \times X$  is defined as:

$$e(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

So, we can say that every Cauchy sequence in  $(X, e)$  is convergent in  $X$ .

Suppose we have a metric  $d$  on  $X$  such that the topology induced by  $d$  is

$$\mathcal{T}_X = \{A \mid A \subseteq X\}.$$

Does this imply that  $d$  is the discrete metric? **No.**

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## Counterexample

Define a metric  $d: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$  as:

$$d(m, n) = \left| \frac{1}{n} - \frac{1}{m} \right|,$$

where  $|\cdot|$  denotes the usual Euclidean metric on  $\mathbb{R}$ .

- Every subset of  $\mathbb{N}$  is open in this metric space. Thus,

$$\mathcal{T}_{\mathbb{N}} = \{A \mid A \subseteq \mathbb{N}\}.$$

• However, this space is **not complete**. Consider the sequence  $x_n = n$ . It is Cauchy but **not** convergent in  $(\mathbb{N}, d)$ .

This counterexample shows that even if the topology induced by a metric is discrete topology, the metric itself need not be the discrete metric.

Moreover, if you are familiar with the concept of **equivalent metrics**, observe that  $(\mathbb{N}, d)$  is homeomorphic to the metric space  $(\mathbb{N}, |\cdot|)$ . Hence, this example illustrates that **completeness is not preserved under homeomorphism**.