Representations of Symmetric Groups: Part 1

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Introduction

In this post, we aim to glean as much as we cam about the characters of the symmetric groups (today we'll be focusing on S_3 and S_4) using simple properties of characters.

First up, S_3!

The most *natural* representation of $G = S_3$ (in fact, this *hardly* seems like a reprsentation at all!) would to be let $\rho : S_3 \to \mathrm{GL}(V)$ be such that it sends $g \in S_3$ to its 3 by 3 permutation matrix and $V = \mathbb{R}^3$. For instance,

$$\rho(123) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \in \mathrm{GL}(\mathbb{R}^3),$$

and

$$\rho(23) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{GL}(\mathbb{R}^3).$$

Unfortunately, this is *not* an *irreducible* one: all the $\rho(g)$'s leave $(a, a, a) \in \mathbb{R}^3$ for a real a *alone*. Hence, $W := \mathrm{span}\{(1, 1, 1)\}$ is an *invariant* subspace of V (an invariant *line*). The projection operator $p : V \to W$ onto W is

$$p(x_1, x_2, x_3) = \frac{x_1 + x_2 + x_3}{3}(1, 1, 1),$$

for all $(x_1, x_2, x_3) \in \mathbb{R}^3$, and

$$W' := \ker(p) = \mathrm{span}\{(1, 0, -1), (0, 1, -1)\} = \{(x, y, -x - y) : x, y \in \mathbb{R}\}.$$

That is, W' is the orthogonal complement of W under the dot product. By definition, $V = W \oplus W'$.

According to Maschke's theorem (or Theorem 1 from Serre's 1977 book), W' is *also* an invariant subspace. We can do a quick spot check: $\rho(g)((-x-y,x,y)) = (-x-y,x,y)$ for $g=(132)$. Setting $(-x-y,x,y) = \alpha(1,0,-1)+\beta(0,1,-1)$, we see that $\alpha=-x-y$ and $\beta=x$, works and so $(-x-y,x,y) \in W'$, as we expect.

The subrepresentation $(\rho|_W, W)$ is thus just the trivial representation: There just isn't much freedom offered by a good 'ol line. However, the degree 2 reprsentation $(\rho|_{W'}, W')$ is more interesting. Notice that $\dim W' = 2$ so $\mathrm{GL}(W')$ can be indentified with 2 by 2 matrices. Fixing $\mathcal{B} = \{(1,0,-1),(0,1,-1)\}$ as a basis for W', and then writing $\rho(g)$ for $g \in G$ as a matrix, gives us:

$$\rho(12) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \rho(13) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \rho(23) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix},$$

and

$$\rho(123) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad \rho(132) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}.$$

Of course, the identity goes to I_2 as usual.

The corresponding character $\chi_{\rho|'_W}$ (recall that the trace does not depend on the choice of the basis of W') is just 0 if the permutation is even and not the identity and -1 is the permutation is odd. Taking into account the identity, we can say $\chi_{\rho|_{W'}}(\sigma) \equiv 0 \mod 2$ if σ is even and $\chi_{\rho|_{W'}}(\sigma) \equiv 1 \mod 2$ is σ is odd.

We call this irreducible character (why is this not reducible?) χ_{standard} and the corrsponding representation the *standard* representation, $(\rho_{\text{standard}}, W)$.

Now, let that unknown third irreducible character be χ: Let $\chi(\sigma)=\alpha$ for transpositions σ and $\chi(\tau)=\beta$ for 3-cycles τ. By the orthogonality of irreducible characters, we know

$$\langle \chi, \chi_{\text{trivial}} \rangle = \frac{1}{6} \sum_{\sigma \in S_3} \chi(\sigma)\overline{\chi_{\text{trivial}}(\sigma)} = 0 \implies 3\alpha + 2\beta = -1,$$

and utilizing the other character we have

$$\langle \chi, \chi_{\text{standard}} \rangle = 0 \implies \beta = 1.$$

Putting these numbers toegther, we get $\chi(e) = \chi(123) = \chi(132) = 1$ and $\chi(12) = \chi(13) = \chi(23) = -1$.

Thus, $\chi(\sigma)$ is simply the sign of σ! We call it the *sign* character: χ_{sign}.

All in all, we now have the character table of S_3!

Onto S_4!

Let's continue our analysis of symmetric group with the next one: S_4. As usual, we have the trivial character: χ_{trivial}, that returns 1 for all $g \in S_4$. In much the same way as last time, we can construct a natural representation for S_4, that assigns a $g \in S_4$ to the corressponding 4 by 4 permutation matrix, as viewed as an element of $\text{GL}(\mathbb{R}^4)$. This won't we irreducible however, as the vectors that have all coordinates equal in $\mathbb{R}^4$ will be invariant under the action of the $\rho(g)$'s. The 3 dimensional complement of this invariant line will be invariant, and that is our standard representation, which character χ_{standard}. Doing the computations, we get $[2,1,1] \to 1$, $[2,2] \to -1$, $[3,1] \to 0$ and $[4] \to= -1$.

It's time to invoke the orthogonality! We still have two unknown characters: χ_1 and χ_2. Using the sum of squares formula, we have
$1^2 + 1^2 + 3^2 + x_1^2 + x_2^2 = |S_4| = 24$, which implies $x_1^2 + x_2^2 = 13$, which forces $x_1 = \chi_1(e) = 3$ and $x_2 = \chi_2(e) = 1$. Leting χ_1 take on values a_1, a_2, a_3 and a_4 and using the three equations:

$$\langle \chi_1, \chi_{\text{trivial}} \rangle = \langle \chi_1, \chi_{\text{standard}} \rangle = \langle \chi_1, \chi_{\text{sign}} \rangle = 0,$$

we get

- $6a_1 + 3a_2 + 8a_3 + 6a_4 = -3$,
- $-6a_1 + 3a_2 + 8a_3 - 6a_4 = -3$,
- $2a_1 - a_2 - 2a_4 = -3$.

Adding the first two equations, $3a_2 + 8a_3 = -3$. Notice that the a's must be integers, so this is linear diophantine equation. Upon solving, we get
$a_2 = -8n - 1$ and $a_3 = 3n$, for $n \in \mathbb{Z}$. Adding the last two equations, we get $2a_3 - 3a_4 = -3$ and so $a_4 = 2n + 1$, and substituting these expressions into the first equation yields $a_1 = -2n - 1$.

We do the same drill with χ_2 (which takes on the values $b_1, \cdots, b_4$) to get that $b_1 = -8m - 6$, $b_2 = 3m + 2$, $b_3 = 3m + 2$ and $b_4 = 2m + 2$ for $m \in \mathbb{Z}$.

Lastly, we have an equation involving both the n's and m's as $\langle \chi_1, \chi_2 \rangle = 0$, which gives us $312mn + 48m + 240n + 48 = 0$. This implies $m = -1$ and $n = 0$ by the SFFT.

This completes our character table for S_4 - just using orthogonality!

Tensor Products

Recall that we have the notion of the tensor product of two representations - a tool that we can use to possibly build χ_1 and χ_2 from χ_{trivial}, χ_{sign} and χ_{standard}. In that direction, we will derrive an expression for the character of a tensor product. But Before that, we look at a slightly different expression for a character of a representation.

Let's fix a basis $\mathcal{B}_V = \{v_1, \cdots, v_n\}$ for V. Recall that we have a corresponding *dual* basis, $\{v_1^*, \cdots, v_n^*\}$ for V^*, the dual space of V: Given a $v \in V$, $v_i^*(v)$ is the coefficient of v_i in the expansion of v in terms of the $\mathcal{B}_V$ basis. Thus, the matrix representation (with respect to $\mathcal{B}_V$) for $\rho_V(g) \in \text{End}_{\mathbb{C}}(V)$ has $(i,j)-$ entry $v_i^*(\rho(g)(v_j))$. Taking the sum of the diagonal entries to get the trace, we have

$$\chi_V(g) = \sum_{i=1}^{n} v_i^*(\rho(g)(v_i)).$$

Now we can work with the character of a tensor product better: Let W be a vector space with basis $\mathcal{B}_W = \{w_1, \cdots, w_m\}$. Then a basis for $V \otimes W$ is $T = \{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$. A corresponding dual basis for $(V \otimes W)^*$ would be $\{(v_i \otimes w_j)^* : 1 \leq i \leq n, 1 \leq j \leq m\}$, where we define $(v_i \otimes w_j)^*(v_k \otimes w_l) = \delta_{ik}\delta_{jl}$ and extend linearly. That is, $(v_i \otimes w_j)^*$ extracts the coefficient of $v_i \otimes w_j$ in the expansion of the input in the basis T. That gives us $(v_i \otimes w_j)^*(z)$ for an *elementary* tensor

$$z = v \otimes w = \left(\sum_{i=1}^{n} a_i v_i\right) \otimes \left(\sum_{j=1}^{m} b_j w_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j (v_i \otimes b_j)$$

is $a_i b_j$, which is also just $v_i^*(v) w_j^*(w)$.

All in all, armed with this new formula for the trace, we have

$$\begin{aligned}
\chi_{V\otimes W}(g) &= \sum_{i,j \in [n]\times[m]} (v_i \otimes w_j)^*(\rho_{V\otimes W}(\sigma)(v_i \otimes w_j)) \\
&= \sum_{i,j \in [n]\times[m]} (v_i \otimes w_j)^*(\rho_V(g)(v_i) \otimes \rho_W(g)(w_j)) \\
&= \sum_{i,j \in [n]\times[m]} v_i^*(\rho_V(g)(v_i)) w_j^*(\rho_W(g)(w_j)) \\
&= \left(\sum_{i=1}^{n} v_i^*(\rho_V(g)(v_i))\right)\left(\sum_{j=1}^{m} w_i^*(\rho_W(g)(w_j))\right) \\
&= \chi_V(g)\chi_W(g).
\end{aligned}$$

So the tensor produt a representations just has the effect of multiplying the corresponding characters! In fact, going back to our character table for S_4, we can see that $\chi_1 = \chi_{\text{sign}}\chi_{\text{standard}}$ - that's another free character for us!