From Integers to Rings and Ideals: An Introduction to Algebraic Number Theory

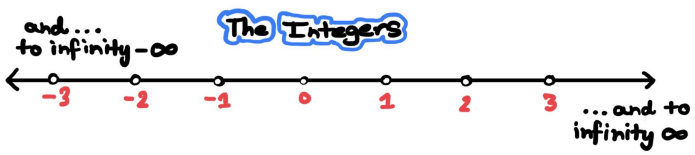
spacersid • 27 Jun 2025

Introduction

One of the key ways mathematics progresses is by first identifying a pattern in a more-or-less *familiar* setting, and then taking a *leap* by trying to extend this pattern to a more *general* environment—one that is, in some sense, *larger* than the previous one. Perhaps the old pattern still holds, or perhaps it doesn't. In the latter case, our goal is to understand exactly *where*, in the leap of abstraction, it fails. Usually, the by-product of this process is the creation of an entirely new landscape of mathematical objects. On the one hand, these objects follow certain intuitive rules inherited from our original setting; on the other hand, they introduce fundamentally new structures. At any rate, the motto here is: *with generalization, through abstraction, comes power*—as we shall explore today.

The Integers: Our First Abstraction

Our story begins with the integers (denoted here by $\mathbb{Z}$), which are essentially the numbers you will recite as you count the hairs on your head or the grains of sand on a beach—along with their negatives. Another way to think about them is to draw them against the backdrop of the continuum of real numbers (or simply a line, which is what the real numbers are!). They appear like a one-dimensional version of galaxies in the universe against the vacuum of space: evenly spaced and discrete, like a sieve. After all, there are *no* integers between, say, 0 and 1, whereas there are an *awful* lot of real numbers—infinite decimals—between 0 and 1! In fact, there are so many real numbers(no matter how far you zoom into the number line, you just keep seeing more) that, we cannot even *count* them—but that is a story for another day.



The Integers: Discrete Like a Sieve.

The integers aren't terribly exciting just sitting there like dots, staring up at us—and they weren't invented to just stay put anyways! Indeed, their $raison\ d'être$ is to do calculations—like after measuring the length and breadth of a plot of land, say 120 meters and 67 meters, we would $multiply$ the two numbers up to get the $area$ of the plot—the number of $tiny$ 1 meter by 1 meter squares that make up the entirety of the $humungous$ plot. And we even developed a fool-proof $algorithm$ to multiply to integers like 120 and 67, something we learnt way back in elementary school! Notice that if I had to figure out, say, the money I would earn if I sold 120 of my hairs for 67 dollars each (yuck!), I would do the $exact$ same thing, even though a huge amount of dirt (that is, the plot) and dead cells (that is, my hair) are nowhere near to being equal!

And just like that, we have our first, albeit $painfully\ trivial$, abstraction, a thing that is strewn in such high quantities across the mathematical landscape that it practically makes it up. The number 120 is a clean 'abstraction', meant to represent 120 hairs and 120 meters of dirt at the same time! We created the integers as a whole by observing a specific commonality between two very distinct sets of objects—their number!

The next step after getting rid of all the $unnecessary\ details$ (In the context of counting, this was simply replacing each instance of that particular object with the same indistinguishable token, so that only differences in the number count, as opposed to a difference in structure or the appearance. At other times, we may be interested in $other$ properties of objects, so that we abstract differently, keep this in mind!) to create a set of 'counting classes', is to formulate meaningful rules to manipulate these 'counting classes'. As we saw, thanks to the reasonably large degree of abstraction that went into creating the integers—in that one integer can stand for a frighteningly large diversity of objects—in the first place, these rules will carry quite a bit of power.

One such rule is multiplication: On the one level, it's just a $procedure$ to convert an input of two integers into a single one (one of infinitely many so called binary operations—try inventing a few yourself!), but it is deeply $grounded$ in a concrete physical meaning—the area of a rectangle—as we saw. The other main rule is that of addition, which has a yet more elementary physical interpretation: the total number of objects when many collections of objects are placed side-by-side. The last step, obviously, is to interpret the result of the abstract manipulation in the context of the situation!

Taking Off

Now, we play around with the integers, building up to our central characters—a very special subset of the integers.

You might notice that given integer 1 and the operation $+$, you can reach any positive integer: $2 = 1 + 1$, $3 = 1 + 1 + 1$ and so on (we're ignoring the negatives and zero for the moment), but that 1 and $\times$ keeps you $firmly$ stuck on 1. Indeed, no $single$ number and $\times$ can create the whole of $\mathbb{Z}^+$ (the set of positive integers). Instead we must broaden our scope and ask what subset S of the integers along with $\times$ can generate $\mathbb{Z}^+$? Clearly we can choose $S = \mathbb{Z}$, so we really should be asking for the $smallest$ possible S. In some sense, S $compresses$ down $\mathbb{Z}$ under multiplication as much as possible!

One way to think about this is to first $restrict$ our field of vision down to the set $\{1, 2\}$, and then increasing it to $\{1, 2, 3\}$ and then to $\{1, 2, 3, 4\}$, continuing on like this, just adding one number at a time, building S for each of these restricted versions of $\mathbb{Z}^+$. You can think about the greatest element of this chain of sets like a $slider$, and we're pulling it out to infinity, a step at a time. We shall use S_n denote the S for $\{1, \cdots, n\}$.

First, for $\{1, 2\}$, S_2 is just the $whole$ thing: you need $both$ 1 and 2, as we noted above. Second, we consider $\{1, 2, 3\}$. There too, you realize that you need 3 as well, as multiplying 2 by itself and by 1 any number of times is $always$ $even$, and 3 is odd: so $3 \notin \{1, 2\}^\times$, where we added that little $\times$ to denote the set of all numbers you can get by multiplying 1 and 2 with each other any number of times. Observe that S_n is always contained in S_{n+1}.

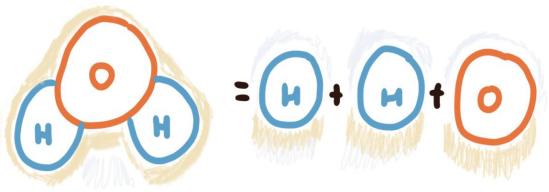
This pattern halts, at least for once, at the third set: $\{1, 2, 3, 4\}$. Here you do not need to add 4 to $S_3 = \{1, 2, 3\}$, as $4 = 2 \times 2$, meaning that $S_4 = \{1, 2, 3\}$. For the first time, we've been able to $truly$ compress such a set, albeit to a very small degree! Try computing more S_n's!

As you might've already guessed, S_n is essentially the set of the so-called $prime\ numbers$: the $building\ blocks$ of the integers under multiplication, or the most compressed version of the integers under multiplication, as we have just seen. The other elements of $\{1, \cdots, n\}$ (the elements of the set theoretic difference $\{1, \cdots, n\} - S_n$), the $compressible$ numbers, are called $composite$ numbers.

A playful analogy to draw here (and one that we will come back to surprisingly often) is to think of the primes as $atoms$ and composite numbers like $molecules$ from chemistry (we are not assigning primes to $quarks$ for a very deliberate

reason!). This analogy has many limitations, one of them being that prime numbers are infinite in number, whereas there are only finitely many stable atoms, but let us overlook these transactional details for the moment.

Indeed, this connection helps us come to terms with a notable property of the integers, that of *unique factorization*. Take the example of a molecule of water —essentially an oxygen atom stuck (the analogue of multiplication) to two hydrogen atoms. Notice that a water molecule will only ever be made up of atoms in this *particular* way: one O atom connected to two H atoms! Drawing on this analogy, it would seem only logical for this to hold in $\mathbb{Z}$ as well: in that every integer can be expressed *uniquely* as the product of primes—the composite number 105 is $3 \times 5 \times 7$ and is *only* $3 \times 5 \times 7$. And, as it turns out, it does actually hold!



Water is, and is always going to be, $\mathrm{H}+\mathrm{H}+\mathrm{O}$.

But this is not always true—only when we are *naive* and *narrow-minded*.

What Next?

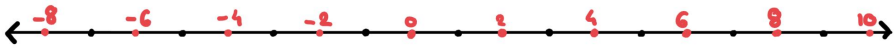
Less abstractly, we're going to move *beyond* the realm of the usual integers, creating *new*, integer *like* objects (mind you, there are *not* the integers, only *similar*), where the *same* element has two *different* prime factorizations—*loss of unique factorization*! These integer like objects will live in the complex *plane*, rather than the real *line*, as you shall soon see. Then, in an attempt to recover unique factorization in this new setting, we will find ourselves replacing individual numbers, with *sets* of numbers—and so we will succeed in creating a new object as a result of *trying* to extend a broken pattern.

Before we set (pun intended!) out on our adventure to foreign landscapes, let us take one last look at divisibility in $\mathbb{Z}$—which will give a hint at those 'sets' we'll ultimately be considering.

Divisibility and Sets

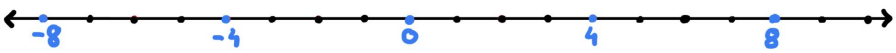
When we say 2 divides 4 (written as $2 \mid 4$) what do we *mean*? On the one hand, it is simply saying that $4/2$ is an integer, as opposed to a fraction or even *worse*, a real number (formally, there exists an integer b such that $4 = 2b$)—but there's also a visual way to analyse this situation.

We start with the integers—evenly spaced and looking up at as usual—and *stretch* on them, keeping 0 fixed, so that the distances between next-door integers increases by a factor of 2: 1 is now where 2 *was* and -4 is where -8 *was*. If we *remove* the labels on the dots and place this transformed line atop the *orginal* one, we see we're left with *2's version of the integers*: the integers you'd be able to vist if you only were armed with a pair of self-regenrating arrows: one that read 'JUMP $+2$' and another that read 'JUMP -2'. In fact, this a way to view the integer 2 as *acting upon* $\mathbb{Z}$: It yields the set—really a proxy of the integers—$2\mathbb{Z}$, the set of all *multiples* of 2.



2's version of the integers.

Now, what happens when we place *4's version of the integers* upon *2's version of the integers*? Didn't notice? Well, try *reversing* the order by *first* placing *4's version of the integers* on the base slate and *then* placing *2's version of the integers* atop that. Did you see what happend? *4's version of the integers* was completely & cleanly *masked* by *2's version of the integers*! Mathematically, the set *4's version of the integers* is contained in the set *2's version of the integers*. This is becuase the new pairs of arrows: one created by gluing two 'JUMP $+2$' arrows and the other created by gluing two 'JUMP -2' arrows means you can visit even *less* integers that before—you've surrendered your precise visting abilities! Essentialy this clean *containment* of sets is manifestation of the fact that 2 *divides* 4—try and see this!



4's version of the integers.

On the other hand, the result is not *so* satisying if we place *3's version of the integers* atop *2's version of the integers* or the the other way. We don't get a clean containment—we can see orange and red dots lurking here and there. This is preceisly becuase 2 does *not* divide 3. Indeed, *n's version of the integers* is always contained in $\mathbb{Z}$—which is *1's version of the integers*—because every integer is divisible by 1.

To sum up, we can view a single integer as *set*—its version of the integers which is its multiples—and divisibility can be seen as a containment of those sets.

And now, we create our new integers.

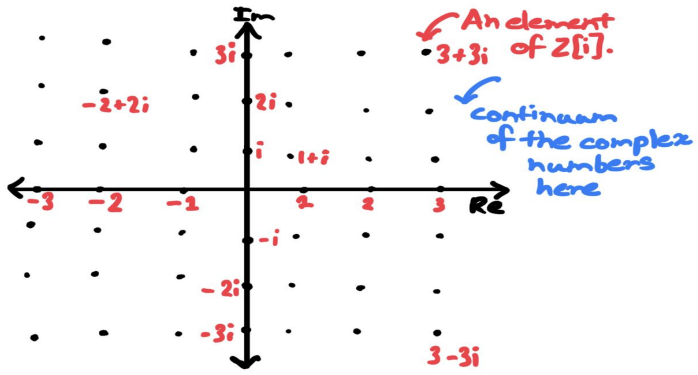
Travelling Beyond

Let's take another look at $\mathbb{Z}$, this time with their *ambient* space in mind—the real numbers, which is essentially a *line*, a one dimensional object. Could we come up with an analogue of the integers for a *plane* rather than a line? That object would truly earn the moniker of a sieve.

Well, mathematically speaking, the complex numbers describe a plane, just like how the real numbers describe a line. As a review, the complex numbers is the set $\{a + bi : a, b \in \mathbb{R}\}$, which can alternatively be viewed as *ordered pairs* of real numbers, along with well-defined rules for addition, subtraction, multiplication and division. The most natural notion of an integer here would be a complex number where both the real and imaginary parts are integers: Geometrically, you're starting with the usual integers, and translating them upwards and downwards in the complex plane by an inetger.

We'll use $\mathbb{Z}[i]$ to denote this notion of the complex integers (notation to be explained!). And sure enough, the picture looks pretty convincing—equally spaced dots looking up at us!

What's more is that one can *still* add, subtract and multiply any two such elements of $\mathbb{Z}[i]$ and get back an element of $\mathbb{Z}[i]$, but you *can't* necesarily divide, in the sense that the quotient of two such elements need not be in $\mathbb{Z}[i]$—*just* like it was in $\mathbb{Z}$! Try and find examples of such quotients! Recall that it was precisely becuase of the fact that you can't cleanly divide *any* two integers it made *sense* or *non-trivial* to talk about divisibility (anything divides into anything in $\mathbb{R}$, anything divides into anything in $\mathbb{C}$).



The complex integers: $\mathbb{Z}[i]$. Equally spaced, like a sieve.

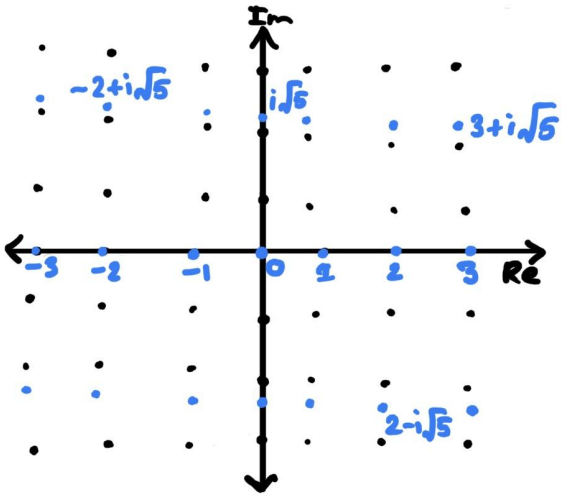
To make things seem a bit *less* abstract, lets actually *write* down the rules for the arithmetic operations on $\mathbb{Z}[i]$. How do we add $a + bi \in \mathbb{Z}[i]$ and $c + di \in \mathbb{Z}[i]$? Just like we add complex numbers! After all, $a + bi$ and $c + di$ are complex numbers—just like how adding two *integers*, say 2 and 3 is the *same* as adding the *real* numbers 2 and 3. Explicitly,

$$(a+bi)+(c+di)=(a+c)+i(b+d).$$

This sum is in $\mathbb{Z}[i]$ because $a+c$, the real part, and $b+d$, the imaginary part are both integers as a, b, c and d are integers and an integer plus another integer is an integer! A very similar analysis can be done on the multiplication!

But this is naive thinking as well. Instead of translating the integers upwards and downwards by an *integer*, why not by an *integeral multiple* of $\sqrt{5}$?

Check it out! This picture *also* looks pretty convincing!



The new version of the complex integers. The elements of this set are colored blue, while the orginal complex integers are black.

Another way to think about it is we've applied the transformation represented by the 2 by 2 matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{5} \end{bmatrix},$$

to each element of $\mathbb{Z}[i]$. We'll denote this set by $\mathbb{Z}[i\sqrt{5}]$ (can you geuss what the notation means?). Written down explicitly, we start with an integer a, and translating it upwards/downwards by an integral multiple of $\sqrt{5}$ is the same thing as adding $ib\sqrt{5}$ for some integer b (the *degree* of translation) to $a-$so $\mathbb{Z}[i\sqrt{5}]=\{a+bi\sqrt{5} : a, b \in \mathbb{Z}\}$. Essentially, we've just multiplied the vector (a, b) with the matrix above.

What's more$-$you can probably geuss what's going to happen$-$we can still add, subtract and multiply elements of $\mathbb{Z}[i\sqrt{5}]$ and get back an element of $\mathbb{Z}[i\sqrt{5}]-$ try verifying this! And then, again, you can't always divide in $\mathbb{Z}[i\sqrt{5}]$, so we can talk about divisibility here as well!

The astute readers among you may not be quite convinced$-$yes, $\mathbb{Z}[i\sqrt{5}]$ is great, but *why* did we have to stretch in the $y-$direction *only*? We could have done the *same* thing with the horizontal axis, streching it out by a factor of $\sqrt{5}$, say?

And we could have completely done that—that set would still **look** discrete (try drawing a picture), and *still* have the basic arithmetic operations *apart* from division. The *only* problem with that would be that we would inadverntantly *loose* 1—the multiplicative identity—in the process, and we *kind* of want that number to be around, as it was present in $\mathbb{Z}$. Stretching in the *vertical* direction allows us to keep 1, lying on the horizontal axis, intact.

Where are the Primes?

Before we can start doing some factorization in these new lands, as promised, we need to *recover* the notion of a prime. What does it mean for an element of $\mathbb{Z}[i]$ or $\mathbb{Z}[i\sqrt{5}]$ to be a prime? To answer that, we ask
$what\ does\ it\ mean\ for\ an\ element\ of\ \mathbb{Z}\ to\ be\ prime?$

Well, it's a usual integer p such that its only divisors are $+1, -1, +p$ and $-p$. Phrased differently, it's a number p such that *if* we write p as a product, say $p = ab$ where a and b are integers, then one of them *must* be $+1$ or -1. Essentially, *only trivial* divisors are allowed to divide that number, not even a *peep* more—and we will be using this exact definition in $\mathbb{Z}[i]$ and $\mathbb{Z}[i\sqrt{5}]$.

Except that there are more trivial divisors—numbers that divide every element—than just $+1$ and -1 here! But what are they?

Let's stick with $\mathbb{Z}[i]$ first. Clearly $+1$ and -1 work, because their *version of* $\mathbb{Z}[i]$—their multiples in $\mathbb{Z}[i]$—is the whole of $\mathbb{Z}[i]$. And why is that? Because they are the *closet* to 0 as possible—their step size, their arrow propelles one forward as little as possible each time it is thrown. In particular, they are a *unit* distance away from the origin. Similarly, $+i$ and $-i$ are also a distance of 1 from the origin, and they divide every complex number as well!

All in all, it can be shown that the four numbers $-1, 1, -i, i$ are the *only* trivial divisors! Any other number will not work because it is *too far*.

There is a way to see this *purely* algebraically as well: we are looking for $a + bi \in \mathbb{Z}[i]$ such that the quotient $(c + di)/(a + bi)$ lies in $\mathbb{Z}[i]$ for *all* $c + di \in \mathbb{Z}[i]$. That is, $a + bi$ divides all $c + di$. Notice that this *necessarily* means that $(a + bi)^{-1}$ *must* be a complex integer, simpy by setting $c + di = 1$. Conversely, *if* $a + bi$ has a multiplicative inverse (that is, $(a + bi)^{-1}$) in $\mathbb{Z}[i]$, then the quotient

$$(c + di)/(a + bi) = (c + di) \times (a + bi)^{-1}$$

is *automatically* in $\mathbb{Z}[i]$ for all $c + di$! An element is a trivial divisor precisely when it is invertible!

Next, we massage the expression for $(a+bi)^{-1}$ a bit, by *real*izing the denominator:

$$\frac{1}{a+bi} = \frac{1}{a+bi} \times \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2}.$$

Recall that both the real and imaginary parts must be integers, so that a^2+b^2 must divide a and also divide b. Try and see if you can complete this argument!

We can do something similar in $\mathbb{Z}[i\sqrt{5}]$. There, the only such numbers are the ol' -1 and $+1$—you might be tempted by $i\sqrt{5}$, but it is *too far* from 0. Indeed, 1 is not even present in $i\sqrt{5}$'s version of the integers as it is *too close* to 0.

We call $-1, 1, i, -i$ to be the *units* of $\mathbb{Z}[i]$, and $-1, 1$ the units of $\mathbb{Z}[i\sqrt{5}]$—named quite naturally! And *now* we can define the analogue of primes in $\mathbb{Z}[i]$: the same definition holds for $\mathbb{Z}[i\sqrt{5}]$.

We say that an element x of $\mathbb{Z}[i]$ is irreducible, if whenever we write x as the product of two elemnts in $\mathbb{Z}[i]$, one of them is a unit.

And now we factorize 6—quite a benign number, to say the least—in $\mathbb{Z}[i\sqrt{5}]$.

The Breaking Point

Well, we all know that $6 = 2 \times 3$. But now we're talking in $\mathbb{Z}[i\sqrt{5}]$, so we have more *exotic* looking factorizations. In fact, notice that $6 = (1+i\sqrt{5})(1-i\sqrt{5})$, and so

$$6 = 2 \times 3 = (1+i\sqrt{5})(1-i\sqrt{5}).$$

This might *hardly* seem surprising. After all, they are *so* many ways to write a regular integer as a product—as a baby example take $24 = 2 \times 12 = 4 \times 6$.

But, in this case, all the involved 6—invited partygoers—$2, 3, 1+i\sqrt{5}, 1-i\sqrt{5}$—are irreducibles—not a casual $4, 6, 12$ or 24 strolling down composite avenue nonchalantly.

'No way!' you shout! How can that be? 2 and 3 may be irreducibles, but $1+i\sqrt{5}$? Seriously? I mean it literally comes in two parts!

But all we have to do is appeal to our freshly baked definition about irreducible—if $1+i\sqrt{5}$ is *really* an irreducible, then *whenever* we write $1+i\sqrt{5} = \alpha\beta$ for some α and β in $\mathbb{Z}[i\sqrt{5}]$, then at least one of them—α or β—*must* be -1 or $+1$, and if that turns out to be true, case closed!

First, let's expand α and β a bit. Leveraging the set-theoretic definition of $\mathbb{Z}[i\sqrt{5}]$, we can write $\alpha = a + bi\sqrt{5}$ and $\beta = c + di\sqrt{5}$, for *some* integers a, b, c and d. Putting everything together,

$$1 + i\sqrt{5} = (a + bi\sqrt{5})(c + di\sqrt{5}).$$

Now, we *could* expand the right side, but that is likely to get *messy*. Instead, we simply take the usual complex number *absolute value* of both sides,

$$\left|1 + i\sqrt{5}\right| = \sqrt{6} = \left|a + bi\sqrt{5}\right| \times \left|c + di\sqrt{5}\right| = \sqrt{a^2 + 5b^2}\sqrt{c^2 + 5d^2}.$$

We don't want those *pesky* square roots, so we simply square both sides to get

$$6 = (a^2 + 5b^2)(c^2 + 5d^2).$$

Notice that both terms on the right side are *usual* integers, and the left side is obvioulsy an integer—and so from an equation in involving elements in $\mathbb{Z}[i\sqrt{5}]$ we havean equation *entirely* in the integers!

Remember *what* we want to do: show that one of $\alpha = a + bi\sqrt{5}$ or $\beta = c + di\sqrt{5}$ *must* be an integer. Well, on the one hand 6 is the product of the two integers $a^2 + 5b^2$ and $c^2 + 5d^2$, and on the other hand, can *only* be written as 6×1 or 2×3 as a product of two (positive) integers! So, $a^2 + 5b^2$ and $c^2 + 5d^2$ *must* be one of $1, 2, 3$ or 6. Let's go case by case!

- If $a^2 + 5b^2$ is 1 then we're done—a must be one and b must be zero, meaning that $\alpha = 1$ (see why this is the case!). Similarly, if $a^2 + 5b^2$ is 6, then the other factor, $c^2 + 5d^2$ must be 1, meaning that $\beta = 1$ this time.
- Next, consider $a^2 + 5b^2 = 3$. This is simply not possible, as $5b^2$ is too large to be 3— it is always greater than 3 for non-zero b, meaning that if this equation were to have a solution, then b must be 0. In that case, $a^2 = \sqrt{3}$, but $\sqrt{3}$ is not even rational, so no integer a exists. Similarly, one can show that $a^2 + 5b^2 = 2$ is not possible.

Phew! That was some algebra heavy-lifting! But at the end of the day, look what we have—$1 + i\sqrt{5}$ is an irreducible in $\mathbb{Z}[i\sqrt{5}]$! And in much the same way, it's partner in crime, $1 - i\sqrt{5}$ is also an irreducible. All in all,

$$6 = \text{Irreducible} \times \text{Irreducible} = \text{Different Irreducible} \times \text{Different Irreducible},$$

something hard to reconcile with our experince in $\mathbb{Z}$!

And with that, we transition into resuce mode. Can we, in *some* way, recover unique factorization in this new set-up?

Try Hard Enough and…

Kummer, a German mathematician, would have none of this corrupt, $\mathbb{Z}[i\sqrt{5}]$—unacceptable business. The way he saw it, we ended up in such a *weird* situation simply because we hadn't factored *enough*.

Taking this thought quite literally, his idea was to have 'numbers'—notice the quotation marks—$\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$ and $\mathfrak{a}_4$ such that $2 = \mathfrak{a}_1 \times \mathfrak{a}_2$ and $3 = \mathfrak{a}_3 \times \mathfrak{a}_4$ on the one side; and on the other side, $1 + i\sqrt{5} = \mathfrak{a}_1 \times \mathfrak{a}_3$ and $1 - i\sqrt{5} = \mathfrak{a}_2 \times \mathfrak{a}_4$.

Now, on the one hand

$$6 = 2 \times 3 = (\mathfrak{a}_1 \times \mathfrak{a}_2) \times (\mathfrak{a}_3 \times \mathfrak{a}_4) = \mathfrak{a}_1 \times \mathfrak{a}_2 \times \mathfrak{a}_3 \times \mathfrak{a}_4,$$

and on the other hand

$$6 = \left(1 + i\sqrt{5}\right) \times \left(1 - i\sqrt{5}\right) = (\mathfrak{a}_1 \times \mathfrak{a}_3) \times (\mathfrak{a}_2 \times \mathfrak{a}_4) = \mathfrak{a}_1 \times \mathfrak{a}_2 \times \mathfrak{a}_3 \times \mathfrak{a}_4,$$

which would avoid all the drama! But what *are* these mysterious characters? Surely they can't be numbers—elememts of $\mathbb{Z}[i\sqrt{5}]$! Indeed: Kummer merely hoped that playing around with these $\mathfrak{a}$'s might lead to some deeper insights.

Let's start by focusing on $\mathfrak{a}_1$. First, $\mathfrak{a}_1 \times \mathfrak{a}_2 = 2$, so $\mathfrak{a}_1$ *divides* 2. Well, then $\mathfrak{a}_1$ must divide *any* multiple of two—just like how $2 \mid 4 \implies 2 \mid 4n$ for any $n \in \mathbb{Z}$. Over here, we have $\mathfrak{a}_1$ divides 2α, where α is in $\mathbb{Z}[i\sqrt{5}]$, as opposed to being *just* in $\mathbb{Z}$. Second, $\mathfrak{a}_1 \times \mathfrak{a}_3 = 1 + i\sqrt{5}$, so $\mathfrak{a}_1$ *also* divides $1 + i\sqrt{5}$, and hence divides $\beta(1 + i\sqrt{5})$, where β is some element of $\mathbb{Z}[i\sqrt{5}]$. Adding the two pieces together, $\mathfrak{a}_1$ divides $2\alpha + \beta(1 + i\sqrt{5})$ (the sum of two multiples of $\mathfrak{a}_1$ is *again* an multiple of $\mathfrak{a}_1$).

Essentially, this means that $2\alpha + \beta(1 + i\sqrt{5})$ for each $\alpha, \beta \in \mathbb{Z}[i\sqrt{5}]$ is *in* $\mathfrak{a}_1$'s version of $\mathbb{Z}[i\sqrt{5}]$! Or, writing it in terms of sets,

$$\left\{2\alpha + \beta(1 + i\sqrt{5}) : \alpha, \beta \in \mathbb{Z}[i\sqrt{5}]\right\} \subseteq \langle \mathfrak{a}_1 \rangle,$$

where we put those funny brackets around $\mathfrak{a}_1$ to denote its version of $\mathbb{Z}[i\sqrt{5}]$—the set of its multiples.

But what is the set on the left *really*?

Well, it is just 2*'s and* $(1 + i\sqrt{5})$*'s version of* $\mathbb{Z}[i\sqrt{5}]$! Using the funny brackets, we write

$$\left\langle 2, 1 + i\sqrt{5} \right\rangle \subseteq \langle \mathfrak{a}_1 \rangle.$$

After a bit of thinking, one realizes that the *containment* $\subseteq$ ought to be an *equality* $=$: Indeed, if $\langle \mathfrak{a}_1 \rangle$ were any *bigger*, while still remaining a version of $\mathbb{Z}[i\sqrt{5}]$, then it would actually be the *whole* of $\mathbb{Z}[i\sqrt{5}]$—try seeing this yourself! That would then imply that $\mathfrak{a}_1$ would be a *unit*: $+1$ or -1, but that not solve our purpose: for then $\mathfrak{a}_2$ would have to be ± 2, which does *not* divide $1 - i\sqrt{5}$ ($(1 + i\sqrt{5})/2$ is not in $\mathbb{Z}[i\sqrt{5}]$), meaning we don't have a factorization at all.

All in all,

$$\langle \mathfrak{a}_1 \rangle = \left\langle 2, 1 + i\sqrt{5} \right\rangle.$$

But, alas, as we already know, there is no *single number* whose version of $\mathbb{Z}[i\sqrt{5}]$—the thing on the right—is *2's and* $(1 + i\sqrt{5})$*'s version of* $\mathbb{Z}[i\sqrt{5}]$—the thing on the left. Try spelling the details out: suppose that $\mathfrak{a}_1$ is really in $\mathbb{Z}[i\sqrt{5}]$. Then use the divisibility relations that $\mathfrak{a}_1$ satisfies to show that it must be either $+1$ or -1—but that would mean that its version of $\mathbb{Z}[i\sqrt{5}]$ is the whole thing, as we just discussed.

So, we must go one step *further*, dropping the bracket: $\langle\rangle$, going from a number of a set, so that

$$\mathfrak{a}_1 = \left\langle 2, 1 + i\sqrt{5} \right\rangle.$$

It is because of the impossibility of writting the $\left\langle 2, 1 + i\sqrt{5} \right\rangle$ *in the form* $\langle \mathfrak{a}_1 \rangle$ *for* $\mathfrak{a}_1$ *in* $\mathbb{Z}[i\sqrt{5}]$ *is why unique factorization fails for* 6*. Thus, the closest we can get is setting* $\mathfrak{a}_1$ *to be the whole set.*

Notice that it is not just 2 that is inside $\mathfrak{a}_1$, but rather the whole of *2's version of* $\mathbb{Z}[i\sqrt{5}]$—or $\langle 2 \rangle$—that is inside $\mathfrak{a}_1$. Recall that $\langle b \rangle \subseteq \langle a \rangle \implies a \mid b$ back in $\mathbb{Z}$. So, we get a fleeting hint that since $\langle 2 \rangle \subseteq \left\langle 2, 1 + i\sqrt{5} \right\rangle$,could we possibily have that the *set* $\left\langle 2, 1 + i\sqrt{5} \right\rangle$ divides the *set* $\langle 2 \rangle$? Well, then we must find $\mathfrak{a}_2$ such that $\left\langle 2, 1 + i\sqrt{5} \right\rangle \times \mathfrak{a}_2 = \langle 2 \rangle$, which is also going to be a *set*! What is $\mathfrak{a}_2$?

Well, in a similar way, one reasons out that $\mathfrak{a}_2 = \left\langle 2, 1 - i\sqrt{5} \right\rangle$ (recall that this is just the set of all possible sums of a multiple of 2 and a multiple of $1 + i\sqrt{5}$ in $\mathbb{Z}[i\sqrt{5}]$).

But then what is

$$\left\langle 2, 1 + i\sqrt{5} \right\rangle \times \left\langle 2, 1 - i\sqrt{5} \right\rangle?$$

What does it *mean* to multiply two sets?

We keep calm and use FOIL. More precisely, we just multiply a random element of the first set with another random element of the second set, and then do this for every pair of elements to get the product set.

Well, on the one hand, an element of the first set may be written as $2\alpha + \beta(1 + i\sqrt{5})$ and an element of the second set may be written as $2\gamma + \delta(1 - i\sqrt{5})$, for some $\alpha, \beta, \gamma, \delta \in \mathbb{Z}[i\sqrt{5}]$. Multiplying everything out, the product is

$$4\alpha\gamma + 2\beta\gamma + 2\beta\delta + 2\alpha\delta - 2i\sqrt{5}(\alpha\delta + \beta\delta - \beta\gamma).$$

Did you notice? Its a $\mathbb{Z}[i\sqrt{5}]$ multiple of 2, since you can cleany factor out a 2! And you can go the other way as well, showing that $every$ multiple of 2 is of this form! In other words,

$$\left\langle 2, 1+i\sqrt{5} \right\rangle \times \left\langle 2, 1-i\sqrt{5} \right\rangle = \langle 2 \rangle,$$

and we have no difficulty in associating the right side with 2. After all, it is $2\text{'s } version \text{ of } \mathbb{Z}[i\sqrt{5}]$! So, while 2 is irreducible as a plain number, it is not irreducible as a version of $\mathbb{Z}[i\sqrt{5}]$.

We can apply the same line of reasoning to 3, to get that

$$\left\langle 3, 1+i\sqrt{5} \right\rangle \times \left\langle 3, 1-i\sqrt{5} \right\rangle = \langle 3 \rangle.$$

Now, watch what happens when we multiply $\left\langle 2, 1+i\sqrt{5} \right\rangle$ and $\left\langle 3, 1+i\sqrt{5} \right\rangle$ – you can try working through the product – we get $\langle 1+i\sqrt{5} \rangle$ and we also have that

$$\left\langle 2, 1-i\sqrt{5} \right\rangle \times \left\langle 3, 1-i\sqrt{5} \right\rangle = \langle 1-i\sqrt{5} \rangle.$$

Along the same lines, while $1+i\sqrt{5}$ is irreducible as a plain number, it is not irreducible as a version of $\mathbb{Z}[i\sqrt{5}]$!

What has this achieved? Well, clearly

$$\langle 6 \rangle = \langle 2 \rangle \times \langle 3 \rangle = \left\langle 2, 1+i\sqrt{5} \right\rangle \times \left\langle 2, 1-i\sqrt{5} \right\rangle \times \left\langle 3, 1+i\sqrt{5} \right\rangle \times \left\langle 3, 1-i\sqrt{5} \right\rangle$$

and on the other side

$$\langle 6 \rangle = \langle 1+i\sqrt{5} \rangle \langle 1-i\sqrt{5} \rangle = \left\langle 2, 1+i\sqrt{5} \right\rangle \times \left\langle 3, 1+i\sqrt{5} \right\rangle \times \left\langle 2, 1-i\sqrt{5} \right\rangle \times \left\langle 3, 1 \right.$$

We have recovered unique factorization! We just had to look beyond numbers and into some special sets.