

Two neat proofs of a sum inequality

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In this short post we are going to study the sum

$$S := \sum_{k=1}^{\infty} \frac{1}{k \cdot \sigma(k)}$$

where σ is a permutation of \mathbb{N} . As you can see, this is a variation of the famous harmonic sum $\sum_{k=1}^{\infty} \frac{1}{k^2}$ whose value is equal to $\frac{\pi^2}{6}$.

If you want to explore this problem by yourself before I reveal more, now is the time. I do believe that fiddling with this sum before reading on would make the rest of this blogpost more interesting.



**Spoiler
below**

The inequality

We are going to prove that

$$\sum_{k=1}^{\infty} \frac{1}{k \cdot \sigma(k)} \leq \frac{\pi^2}{6}$$

Again, if you want to find a proof by yourself, now is the time.

Spoiler below

Proof number 1

We are going to use **Cauchy-Schwarz inequality**. To do so, consider the infinite vectors $(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots)$ and $(\frac{1}{\sigma(1)}, \frac{1}{\sigma(2)}, \frac{1}{\sigma(3)}, \dots)$, and note that their scalar product is exactly S . By Cauchy-Schwarz inequality we have

$$S \leq \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} \cdot \sqrt{\sum_{k=1}^{\infty} \frac{1}{\sigma(k)^2}}$$

And since $\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{\sigma(k)^2} = \frac{\pi^2}{6}$ we can directly deduce the inequality we wanted to prove.

Proof number 2

For this proof, we are going to use the **AM-GM inequality**. For every $k \in \mathbb{N}$ we have that $\frac{1}{k \cdot \sigma(k)} = \sqrt{\frac{1}{k^2} \cdot \frac{1}{\sigma(k)^2}}$ and thus, by the AM-GM inequality, we obtain

$$\frac{1}{k \cdot \sigma(k)} \leq \frac{1}{2} \left(\frac{1}{k^2} + \frac{1}{\sigma(k)^2} \right)$$

By summing all these terms for $k \geq 1$ we obtain $S \leq \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{1}{k^2} + \frac{1}{\sigma(k)^2} \right)$ and since $\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{\sigma(k)^2} = \frac{\pi^2}{6}$ we can directly deduce the inequality we wanted to prove.