Two neat proofs of a sum inequality

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In this short post we are going to study the sum

$$S := \sum_{k=1}^{\infty} \frac{1}{k \cdot \sigma(k)}$$

where σ is a permutation of $\mathbb{N}$. As you can see, this a variation of the famous harmonic sum $\sum_{k=1}^{\infty} \frac{1}{k^2}$ whose value is equal to $\frac{\pi^2}{6}$.

If you want to explore this problem by yourself before I reveal more, now is the time. I do believe that fiddling with this sum before reading on would make the rest of this blogpost more interesting.

The inequality

We are going to prove that

$$\sum_{k=1}^{\infty} \frac{1}{k \cdot \sigma(k)} \leqslant \frac{\pi^2}{6}$$

Again, if you want to find a proof by yourself, now is the time.

Spoiler below

Proof number 1

We are going to use Cauchy-Schwarz inequality. To do so, consider the infinite vectors $\left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\right)$ and $\left(\frac{1}{\sigma(1)}, \frac{1}{\sigma(2)}, \frac{1}{\sigma(3)}, \ldots\right)$, and note that their scalar product is exactly S. By Cauchy-Schwarz inequality we have

$$S \leqslant \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} \cdot \sqrt{\sum_{k=1}^{\infty} \frac{1}{\sigma(k)^2}}$$

And since $\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{\sigma(k)^2} = \frac{\pi^2}{6}$ we can directly deduce the inequality we wanted to prove.

Proof number 2

For this proof, we are going to use the AM-GM inequality. For every $k \in \mathbb{N}$ we have that $\frac{1}{k \cdot \sigma(k)} = \sqrt{\frac{1}{k^2} \cdot \frac{1}{\sigma(k)^2}}$ and thus, by the AM-GM inequality, we obtain

$$\frac{1}{k \cdot \sigma(k)} \leqslant \frac{1}{2}\left(\frac{1}{k^2} + \frac{1}{\sigma(k)^2}\right)$$

By summing all these terms for $k \geqslant 1$ we obtain $S \leqslant \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{k^2}+\frac{1}{\sigma(k)^2}\right)$ and since $\sum_{k=1}^{\infty}\frac{1}{k^2} = \sum_{k=1}^{\infty}\frac{1}{\sigma(k)^2} = \frac{\pi^2}{6}$ we can directly deduce the inequality we wanted to prove.