A simple solution to the painter’s paradox

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I have seen many unsatisfactory explanations for solving this paradox, and I just wanted to provide a (hopefully) clear one here. Though a great explanation is given in the book *Paradoxes and Sophisms in Calculus* (p. 64, here in .pdf) by Sergiy Klymchuk and Susan Staples. I haven’t read the rest of the book, but based on this little snippet I am sure it is really nice.

Gabriel’s horn

Gabriel’s horn (or Torricelli’s trumpet) is a mathematical surface which possesses the following properties: - It is a smooth surface whose area is infinite - The volume it defines is finite It is constructed by considering the inverse function $x \mapsto \frac{1}{x}$ on the domain $[0, +\infty[$, and rotating it around the x-axis. Here is an illustration.

Gabriel’s horn

Gabriel’s horn

The painter’s paradox

Because of its properties, the trumpet seems to give rise to a paradox. Since it defines a finite volume, then we can fill it with a finite amount of paint that would coat the entire surface, even though the latter is infinite.

Of course here we consider mathematical paint. It can be spread as thin as we wish (though the thickness cannot be 0), and we ignore the action of filling the horn or the time it takes. We suppose it is already filled.

Resolution of the paradox

The apparent paradox lies in our belief that an infinite surface requires an infinite amount of paint to be covered entirely. We think this because we implicitly consider a uniform coating of paint. However, this is not the case when we fill the horn with paint, as the tube becomes arbitrarily narrow the further down we go.

In fact, for any finite amount of paint, we are able to paint any surface of infinite area. To do so, use half of your paint to cover a finite part of the surface. Then use half of what remains to cover another part of the surface of the same area, by spreading the paint twice as thin. Repeat this process infinitely.