Awesome Math Test C Part I Question 2

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Awesome Math Test C Part I Question 2:

$$\frac{1}{2}(a^3 + \frac{1}{a^3})^2 = 2025 \tag{1}$$

Evaluate

$$\frac{1}{2}(a + \frac{1}{a})^2 \tag{2}$$

The first thing that we want to do is to figure out what connections we could find by comparing euqation (1) and expression (2). We see that the left hand side of equation (1) is farily similar to expression (2), so we could try to isolate the terms that contains a from both equation (1) and expression (2).

$$\begin{aligned}
\frac{1}{2}(a^3 + \frac{1}{a^3})^2 &= 2025 \\
(a^3 + \frac{1}{a^3})^2 &= 2 * 2025 \\
a^3 + \frac{1}{a^3} &= \sqrt{2 * 2025} \\
a^3 + \frac{1}{a^3} &= 45\sqrt{2} \qquad (3)
\end{aligned}$$

For expression (2), let us set the result of the expression to x, which will tern it into an equation solving for x

$$\begin{aligned}
\frac{1}{2}(a + \frac{1}{a})^2 &= x \\
(a + \frac{1}{a})^2 &= 2x \\
a + \frac{1}{a} &= \sqrt{2x} \qquad (4)
\end{aligned}$$

Now we see that equation (3) and equation (4) is very similar, we could cube equation (4) in order to make the substitution of x^3 and $\frac{1}{x^3}$.

$$\begin{aligned}
a + \frac{1}{a} &= \sqrt{2x} \\
(a + \frac{1}{a})^3 &= (\sqrt{2x})^3
\end{aligned}$$

$$a^3 + 3a^2\frac{1}{a} + 3a\frac{1}{a}^2 + \frac{1}{a}^3 = (\sqrt{2x})^3$$
$$a^3 + 3a + 3\frac{1}{a} + \frac{1}{a}^3 = (\sqrt{2x})^3$$

Now we substitute in for $a^3 + \frac{1}{a^3} = 45\sqrt{2}$ from equation (3)

$$a^3 + 3a + 3\frac{1}{a} + \frac{1}{a}^3 = (\sqrt{2x})^3$$
$$3a + 3\frac{1}{a} + 45\sqrt{2} = (\sqrt{2x})^3$$
$$3(a + \frac{1}{a}) + 45\sqrt{2} = (\sqrt{2x})^3$$

We could also substitute $a + \frac{1}{a} = \sqrt{2x}$ from equation (4)

$$3(a + \frac{1}{a}) + 45\sqrt{2} = (\sqrt{2x})^3$$
$$3(a + \frac{1}{a}) + 45\sqrt{2} = (a + \frac{1}{a})^3 \tag{5}$$

Now we could substitute $y = a + \frac{1}{a}$ for equation (5) and came up with a cubic function.

$$3y + 45\sqrt{2} = y^3$$
$$y^3 - 3y - 45\sqrt{2} = 0 \tag{6}$$

Upon looking at equation, because of how there is a $\sqrt{2}$, we could guess that &y& will also include a $\sqrt{2}$, so we could substitute $y = k\sqrt{2}$.

$$(k\sqrt{2})^3 - 3(k\sqrt{2}) - 45\sqrt{2} = 0$$
$$(k\sqrt{2})^3 - 3(k\sqrt{2}) = 45\sqrt{2}$$
$$k^3(2\sqrt{2}) - 3(k\sqrt{2}) = 45\sqrt{2}$$
$$(2k^3 - 3k)(\sqrt{2}) = 45\sqrt{2}$$
$$2k^3 - 3k - 45 = 0$$

Now, because we have a cubic equation with such small numbers for its coeffecient, we could do some testing. We would be able to figure out that $k = 3$ is a solution, so substitution $k = 3$ for $y = k\sqrt{2}$, we get $y = 3\sqrt{2}$. Further substitution $y = a + \frac{1}{a}$, we have $a + \frac{1}{a} = 3\sqrt{2}$. Now we could just compute and substitute it into expression (2).

$$\frac{1}{2}(a + \frac{1}{a})^2$$

$$\frac{1}{2} * (3\sqrt{2})^2$$

$$\frac{1}{2} * 18$$

$$9$$

We've solved it! The answer to this question is 9!