Equivalence of Standard and Twisted Special Orthogonal Lie Algebras

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Let n be a positive integer. For a non-degenerate symmetric matrix $S \in \mathfrak{gl}_n(\mathbf{C})$, define

$$\mathfrak{o}_n(\mathbf{C}, S) := \{X \in \mathfrak{gl}_n(\mathbf{C}) : X^T S = -SX\},$$

called the *orthogonal Lie algebra* with respect to S. The standard definition of the *special orthogonal Lie algebra* is given by

$$\mathfrak{so}_n(\mathbf{C}) := \mathfrak{o}_n(\mathbf{C}, I_n),$$

where "special" implicitly indicates the fact that $\mathfrak{so}_n(\mathbf{C}) \subseteq \mathfrak{sl}_n(\mathbf{C})$.

There is another common definition of the special orthogonal Lie algebra, which we refer to here as the *twisted special orthogonal Lie algebra*, defined as the Lie algebra

$$\mathfrak{o}_n(\mathbf{C}, S), \quad S = \begin{cases} \begin{bmatrix} 0 & I_k \\ I_k & 0 \end{bmatrix} & \text{if } n = 2k \text{ is even,} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ I_n & 0 & 0 \end{bmatrix} & \text{if } n = 2k+1 \text{ is odd.} \end{cases}$$

The following theorem shows that these two definitions are equivalent.

Theorem Let n be a positive integer. Let $S, T \in \mathfrak{gl}_n(\mathbf{C})$ be two non-degenerate symmetric matrices. Then there exists an invertible matrix $P \in \mathfrak{gl}_n(\mathbf{C})$ such that $P^T SP = T$. Under this transformation,

$$\mathfrak{o}_n(\mathbf{C}, S) \to \mathfrak{o}_n(\mathbf{C}, T), \quad X \mapsto P^{-1}XP$$

is an isomorphism of Lie algebras.

The existence of P in this theorem follows from the fact that any two non-degenerate symmetric matrices are congruent over $\mathbf{C}$.

The standard definition $\mathfrak{so}_n(\mathbf{C})$ has the benefit of being simpler, and its simplicity (for $n \geq 3$) is easier to derive (by brute force calculation). However, the Cartan subalgebra and its Cartan decomposition are very difficult to describe:

Exercise Let n be a positive integer. Prove that

$$\text{span}\{E_{12}, E_{34}, \ldots, E_{k-1,k}\}, \quad k := \lfloor \frac{n}{2} \rfloor$$

is a Cartan subalgebra of $\mathfrak{so}_n(\mathbf{C})$, where $E_{ij} = e_{ij} - e_{ji}$.

Exercise

Let n be a positive integer. Let H be the Cartan subalgebra of $\mathfrak{so}_n(\mathbf{C})$ defined as above.

1. If $n = 2m$ is even, prove that a Cartan decomposition of $\mathfrak{so}_n(\mathbf{C})$ is

$$H \oplus \bigoplus_{k,l} \operatorname{span} \left\{ \begin{matrix} (E_{k-1,l-1} + iE_{k-1,l}) + i(E_{k,l-1} + iE_{k,l}), \\ (E_{k-1,l-1} - iE_{k-1,l}) + i(E_{k,l-1} - iE_{k,l}), \\ (E_{k-1,l-1} + iE_{k-1,l}) - i(E_{k,l-1} + iE_{k,l}), \\ (E_{k-1,l-1} - iE_{k-1,l}) - i(E_{k,l-1} - iE_{k,l}) \end{matrix} \right\},$$

   where k, l runs over all even integers such that $1 \leq k < l \leq n$.

2. If $n = 2m + 1$ is odd, prove that a Cartan decomposition of $\mathfrak{so}_n(\mathbf{C})$ is

$$H \oplus \bigoplus_{k,l} \operatorname{span} \left\{ \begin{matrix} (E_{k-1,l-1} + iE_{k-1,l}) + i(E_{k,l-1} + iE_{k,l}), \\ (E_{k-1,l-1} - iE_{k-1,l}) + i(E_{k,l-1} - iE_{k,l}), \\ (E_{k-1,l-1} + iE_{k-1,l}) - i(E_{k,l-1} + iE_{k,l}), \\ (E_{k-1,l-1} - iE_{k-1,l}) - i(E_{k,l-1} - iE_{k,l}) \end{matrix} \right\} \oplus \bigoplus_{m} \operatorname{span} \left\{ \begin{matrix} (E_{m-1,n} + iE_{m,n}), \\ (E_{m-1,n} - iE_{m,n}) \end{matrix} \right\},$$

   where k, l runs over all even integers such that $1 \leq k < l \leq n$, and m runs over all even integers such that $1 \leq m \leq n$.

The twisted definition $\mathfrak{o}_n(\mathbf{C}, S)$ has the advantage that the diagonal matrices in $\mathfrak{o}_n(\mathbf{C}, S)$ form its Cartan subalgebra H, and the standard basis of $\mathfrak{o}_n(\mathbf{C}, S)$ coincides with its weights with respect to H.