Normalizer and Generalized Weightspaces in Lie Algebras

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Exercise
Let L be a complex finite dimensional Lie algebra and M a nilpotent subalgebra of L. Regard L as an M-module.

1. Prove that $N_L(M) \subseteq L_0$.
2. Prove that $N_L(M) = L_0$ if and only if M is an ideal of L_0.
3. Find an example of M such that M is not a Cartan subalgebra and $N_L(M) = L_0$.

Recall that L_0 is the generalized weightspace with respect to the weight 0, so for an element $x \in L$, $x \in L_0$ if and only if for every $m \in M$, there is a positive integer n such that $\mathrm{ad}_m^n(x) = 0$.

To prove (1), let $x \in N_L(M)$ and $m \in M$. Then $[M, x] \subseteq M$. Hence $[m, x] \in M$. Since M is finite-dimensional and nilpotent, there is a positive integer n such that $M^n = 0$. Hence

$$\mathrm{ad}_m^n(x) = \mathrm{ad}_m^{n-1}([m, x]) \subseteq \mathrm{ad}_m^{n-1}(M) \subseteq M^n = 0.$$

Thus $x \in L_0$, as desired.

To prove (2), note that by definition, M is an ideal of L_0 if and only if $N_{L_0}(M) = L_0$, where $N_{L_0}(M) = N_L(M) \cap L_0$. Since $N_L(M) \subseteq L_0$ by (1), we have $N_{L_0}(M) = N_L(M)$. Hence M is an ideal of L_0 if and only if $N_L(M) = L_0$, as desired.

For (3), choose L to be the unique 2-dimensional complex Lie algebra with a basis $\{x, y\}$ such that $[x, y] = x$. Then $M = \mathbf{C}x$ is a nilpotent subalgebra of L such that $N_L(M) = L$. Hence M is not a Cartan subalgebra, and we have $N_L(M) = L_0$ by (1).