An Example of a Maximal Nilpotent Subalgebra that is not a Cartan Subalgebra

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We start with an easy exercise.

Exercise Let L be the 3-dimensional complex Lie algebra with basis $\{e_1, e_2, e_3\}$ such that

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$

Find a Cartan decomposition of L. $\diamondsuit$

Note that this Lie algebra is not nilpotent, because L is not ad-nilpotent:

$$[\mathrm{ad}_{e_1}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\mathrm{ad}_{e_2}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\mathrm{ad}_{e_3}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It is clear that $M = \mathrm{span}\{e_1, e_2\}$ is a nilpotent Lie subalgebra of L of dimension 2, and since L is not nilpotent, we see that M is a maximal nilpotent subalgebra of L. In fact, M is a maximal abelian subalgebra of L. However, M is not a Cartan subalgebra. For example, note that M is not self-normalized, because $[M, e_3] \subseteq M$ but $e_3 \notin M$.

A Cartan subalgebra of L can be chosen, for example, $H = \mathrm{span}\{e_3\}$, as it is easy to see that H is self-normalized, and is of course nilpotent. It turns out that M and H are both maximal nilpotent subalgebra, and H has smaller dimension than M.

A Cartan decomposition of L is

$$L = \mathrm{span}\{e_3\} \oplus \mathrm{span}\{e_1, e_2\} = L_0 \oplus L_{-1}.$$

Although M is not a Cartan subalgebra, we can still find the weight space decomposition of L as M acts on L. Since e_1 and e_2 are ad-nilpotent, the weight space decomposition is just

$$L = \mathrm{span}\{e_1, e_2, e_3\} = V_0.$$

Note that this generalized weight space V_0 (with respect to weight 0) is strictly larger than M, which is not strange because M is not a Cartan subalgebra.