The subobject classifier in the category of representations of a monoid

written by rapha on Functor Network
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The case when the monoid is a group

Let G be a group (written multiplicatively). We are interested in $\mathbf{B}G$, the category of (set) representations of G. Objects in this category are pairs (X, μ) where X is a set and $\mu : X \times G \to X$ is a right action of G on X — let's immediately agree to simply write $x \cdot g$ instead of $\mu(x, g)$. Recall that an action must verify the axioms:

- $x \cdot 1 = x$; and
- $x \cdot (gh) = (x \cdot g) \cdot h$.

Arrows in this category are set functions between the underlying sets $f : X \to Y$ that respect the group action: for all $g \in G$, we must have $f(x \cdot g) = f(x) \cdot g$. Composition of arrows is just the usual composition of the underlying functions.

Let 1 denote the singleton set, and 2 the set with two elements, both with the trivial action $x \cdot g = x$. For any set X, the unique function $X \to 1$ respects the group action. Hence, the representation 1 is the terminal object in $\mathbf{B}G$. We want to show that the representation morphism $\top : 1 \rightarrowtail 2$ induced by the inclusion $1 = \{\top\} \subseteq \{\top, \bot\} = 2$ is the subobject classifier in the category $\mathbf{B}G$.

Notice also that the pullback of any diagram $X \xrightarrow{f} B \xleftarrow{g} Y$ always exists in $\mathbf{B}G$, and is given by the usual pullback in $\mathbf{Set}$ with the obvious coordinate-wise action. More precisely,

$$X \times_B Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\},$$

with the action $(x, y) \cdot g$ defined to be $(x \cdot g, y \cdot g)$. The existence of a terminal object and all pullbacks is sufficient for (in fact, equivalent to) having all finite limits in $\mathbf{B}G$, and these may be constructed as limits of the carrier sets. In other words, the forgetful functor $U : \mathbf{B}G \to \mathbf{Set}$ preserves all finite limits.

Let $m : S \rightarrowtail X$ be a (representative for a) subobject of X. The group G acts on itself by multiplication on the right, and for any fixed element $s \in S$, we have a morphism of representations $\lambda_s : G \to S$ defined as $\lambda_s(g) = s \cdot g$. We can leverage these morphisms to prove the function underlying m must be injective. Let $s, t \in S$ be two elements of S such that $m(s) = m(t)$. This relation, together with the fact m preserves the group action, means that we have $m \circ \lambda_s = m \circ \lambda_t$. Hence $\lambda_s = \lambda_t$. In particular, equality holds when these morphisms are evaluated at the identity element of G, so $s = t$. Since, as we've just shown, the function underlying any subobject must be injective, each subobject has a representative

$S \rightarrowtail X$ where S is a *subset* of X, with the group action on S being the group action on X restricted to S. Therefore, any subobject of X in $\mathbf{B}G$ is (represented by) a subset S of X which is stable by the group action, i.e. for all $s \in S$ and for all $g \in G$, we have $s \cdot g \in S$.

Given a subrepresentation $S \subseteq X$, the characteristic set function $\chi_S : X \to 2$, defined as $\chi_S(x) = \top$ if and only if $x \in S$, respects the group actions. Indeed, suppose $x \in S$; then, for all $g \in G$, we have $\chi_S(x \cdot g) = \top$ because S is closed under the action of G, while $\chi_S(x) \cdot g = \chi_S(x) = \top$ because the action on 2 is trivial. On the other hand, the complement of S in X is *also* a subrepresentation because G is a group, so when $x \notin S$, we also have $x \cdot g \notin S$ for all $g \in G$, whence $\chi_S(x \cdot g) = \bot = \chi_S(x) \cdot g$ in this case.

Because $S = \chi_S^{-1}(\top)$, it follows that this is a pullback square:

$$\begin{array}{ccc} S & \xrightarrow{\quad ! \quad} & 1 \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{\quad \chi_S \quad} & 2 \end{array}$$

The only thing left to show for $1 \hookrightarrow 2$ to be a subobject classifier is that χ_S is the *only* representation morphism such that the previous diagram is a pullback diagram. Suppose $\phi : X \to 2$ is another one. Then $S = \phi^{-1}(\top)$, so that $\phi(x) = \top$ if and only if $\chi_S(x) = \top$. Since there are only two possible values that these functions can take, we must have $\chi_S = \phi$.

The previous discussion could have been made perhaps clearer by the use of the forgetful functor $U : \mathbf{B}G \to \mathbf{Set}$, which, as we've seen, preserves limits. For instance, pick some monic arrow $m : S \rightarrowtail X$. An arrow is monic if and only if $(1_S, 1_S)$ is a pullback for the pair (m, m). Therefore any functor which preserves limits also preserves monic arrows. In our particular case, we see that $U(m)$ is a monic arrow in **Set**. It is known that monic arrows in **Set** are precisely the injective functions. Since $U(m)$ is the set function underlying the morphism of representations m, we must have that m is injective, as we've shown in a more roundabout way a couple of paragraphs ago. The uniqueness of the characteristic function χ_S may also be proven quickly using the functor U: the image of any pullback diagram in $\mathbf{B}G$ through U is also a pullback diagram in **Set**, and we already know that the characteristic function is the unique set function such that $U(m)$ is the pullback of $U(1) \to U(2)$ along it.

The general case

Most of our previous discussion carries through when we consider $G = M$ to be simply a monoid (not necessarily a group). We are interested in $\mathbf{B}M$, the category of (set) representations of the monoid M. As before, subobjects of X are identified with subsets $S \subseteq X$ that are stable under the monoid action, and are called "subrepresentations". The same arguments also give that 1 with the trivial action is the terminal object in $\mathbf{B}M$, and the pullback of any pair of arrows always exists (the forgetful functor still preserves finite limits).

In fact, from our previous discussion of group representations, there is only one thing that changes: the characteristic function χ_S is not necessarily a morphism of representations anymore. This breaks a lot of stuff, and we don't have such an easy time finding a subobject classifier. The fundamental reason of why the characteristic function doesn't respect the monoid action, is that the complement of a subrepresentation may not be a subrepresentation: in the group case, the complement was always a stable under the action, but for monoids in general this is not true. A somewhat artificial example is given by the action of the monoid $(\mathbb{N}, +)$ on the set of relative integers $\mathbb{Z}$ in the obvious way: for all $z \in \mathbb{Z}$ and all $n \in \mathbb{N}$, we define $z \cdot n$ to be $z + n$. Then, for any $n \geq 0$, the set of all integers in $\mathbb{Z}$ greater or equal to n is a subrepresentation, but the complement of such a set is never a subrepresentation.

Let's introduce some terminology. We say that a subset I of M is an **ideal** if, for every $i \in I$ and every $m \in M$, we have $im \in I$. Let X be a representation of M, and let S be a subrepresentation of X. We say that an element $m \in M$ **kills** an element $x \in X$ (relative to S) when we have $x \cdot m \in S$. For any element $x \in X$, let $I_S(x)$ be the set of elements in M which kill x relative to S. This set $I_S(x)$ is an ideal in M. Notice that for all $s \in S$, we have $I_S(s) = M$; moreover, when M is a group, we have $I_S(x) = \varnothing$ for every $x \notin S$. In fact, when M is a group, there are only two ideals in M: the empty set $\varnothing$ and the whole group M. However, when M is a monoid, it is not true that $I_S(x) = \varnothing$ for all $x \notin S$. For instance, in our previous artificial example,

$$I_{\mathbb{N}}(-2) = \{n \in \mathbb{N} \mid n \geq 2\}.$$

We define Ω to be the set of all ideals of M. In our notation, for any subrepresentation S of X, we have a function $I_S : X \to \Omega$. We wish to define a monoid action on Ω so that I_S is a morphism of representations. In other words, we want the ideal $I_S(x \cdot m)$ of elements that kill $x \cdot m$ to be precisely $I_S(x) \cdot m$. This forces us to define, for all ideals $I \subseteq M$ and all $m \in M$, the action

$$I \cdot m = \{k \in M \mid mk \in I\}.$$

This action makes Ω into an object of $\mathbf{B}M$ and each function I_S becomes a morphism of representations. When M is a group, Ω reduces to the two-elements set and I_S reduces to the characteristic function χ_S. To continue this analogy further, we define $\top : 1 \rightarrowtail \Omega$ by sending the unique element of 1 to the top element $M \in \Omega$.

Because $x \in S$ if and only if $I_S(x) = M$, we see that $S = I_S^{-1}(M)$. Therefore, the following diagram is a pullback square:

$$\begin{array}{ccc} S & \xrightarrow{\quad ! \quad} & 1 \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{\quad I_S \quad} & \Omega \end{array}$$

Now we need to show I_S is the *only* arrow such that the previous diagram is a pullback square. Suppose $\phi : X \to \Omega$ is another one, and pick any $x \in X$. We want to show that $I_S(x) = \phi(x)$.

Let m be an element of $I_S(x)$, so that $x \cdot m \in S$. By the commutativity of the previous diagram with ϕ in place of I_S, this means $\phi(x \cdot m) = M$. Since ϕ respects the monoid action, we have $\phi(x) \cdot m = M$. Hence, by the definition of the monoid action on Ω, we have $\{k \in M \mid mk \in \phi(x)\} = M$. In particular, since $1 \in M$, we have $m \in \phi(x)$. Because m was an arbitrary element of $I_S(x)$, we find that $I_S(x) \subseteq \phi(x)$.

Consider $T = x \cdot \phi(x)$ the subset of X defined as the set of elements of the form $x \cdot i$ with $i \in \phi(x)$. Since $\phi(x)$ is an ideal of M, this subset T is stable under the monoid action and thus is a subrepresentation of X. Moreover, for any $t \in T$, we have $\phi(t) = M$. Indeed, any t may be written in the form $x \cdot i$ for some $i \in \phi(x)$; then, $\phi(t)$ $(= \phi(x) \cdot i)$ is by definition the set of all $k \in M$ such that $ik \in \phi(x)$, and this condition is always verified since $\phi(x)$ is an ideal. Thus the universal property of the pullback gives us an inclusion $T \subseteq S$. In particular, all elements $m \in \phi(x)$ have the property that $x \cdot m \in S$, so that $\phi(x) \subseteq I_S(x)$.

The two previous paragraphs prove, by double inclusion, that $I_S(x) = \phi(x)$. Because x was an arbitrary element of X, this shows I_S is the unique function such that S is the pullback of $1 \to \Omega$ along I_S. Therefore, $1 \to \Omega$, defined by sending the unique element of 1 to the top element $M \in \Omega$, is the subobject classifier in the category $\mathbf{B}M$ of representations of a fixed monoid M.