On subobject classifiers

written by rapha on Functor Network
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I'm currently reading Sheaves in Geometry and Logic, by MacLane and Moerdijk. Possibly, subobject classifiers are going to be important, so here are some notes to make sure I understand this stuff correctly. The text of the book is pretty clear, but a lot of details are glossed over.

Recall that in any category, a **subobject** of some object X is an equivalence class of monic arrows $S \rightarrowtail X$, where two such arrows are said to be equivalent if and only if there is an isomorphism between their domain that makes the obvious triangle commute. Notice that because we're talking about monic arrows, if there is such an isomorphism making the triangle commute, then that isomorphism is necessarily the *only one* making the triangle commute.

Now suppose our ambient category has all finite limits (including the limit of the empty diagram, that is, the terminal object 1). In this context only, a **subobject classifier** is a monic arrow $\top : 1 \rightarrowtail \Omega$ such that, for any monic arrow $m : S \rightarrowtail X$, there exists a unique $\phi_S : X \to \Omega$ which makes the following diagram into pullback diagram (cartesian square):

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle \top} \\ X & \xrightarrow{\phi_S} & \Omega \end{array}$$

An important thing to notice here is that the "characteristic function" ϕ_S must actually be the *same* arrow for all representatives of a single subobject. Indeed, suppose $m' : S' \rightarrowtail X$ is another monic such that there exists an isomorphism $\alpha : S' \to S$ such that $m' = m \circ \alpha$ (that is, suppose m and m' are both representatives of the same subobject of X). There exists a $\phi_{S'}$ having the pullback square property of the previous definition with respect to S'. Now one can show that

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle \top} \\ X & \xrightarrow{\phi_{S'}} & \Omega \end{array}$$

(caution: we use $\phi_{S'}$ as the lower morphism) is a pullback square. Hence by the unicity clause of the definition, we must have $\phi_S = \phi_{S'}$.

Recall that $\mathrm{Sub}(X)$ is the class of all subobject of X. By the preceeding remark, we have a well-defined function

$$\theta_X : \mathrm{Sub}(X) \to \mathrm{Hom}(X, \Omega)$$

which sends a subobject S to its "characteristic function" ϕ_S. In fact, this is a bijection! It is a surjection because the pullback of $\top$ (or more generally any monic) along any morphism $X \to \Omega$ is a monic arrow, and so represents a subobject of X. It is an injection because any two pullbacks of $X \to \Omega \leftarrowtail 1$ are isomorphic. Consequently, if the ambient category is locally small, then it is also well-powered (recall this means $\mathrm{Sub}(X)$ is a set for each object X). With the category being locally small, we have even more: the collection of all bijections θ_X assemble into a natural isomorphism of (contravariant) functors

$$\mathrm{Sub}(-) \cong \mathrm{Hom}(-, \Omega).$$

In other words, the functor Sub is representable. Let's prove this claim. Recall that the action of the functor Sub on morphisms $f : Y \to X$ is by "pulling back": the arrow $\mathrm{Sub}(f) : \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ is defined to be the set function which sends a subobject (represented by) $m : S \rightarrowtail X$ to the subobject represented by m', the pullback of m along f. This is well-defined (independant of the choice of representative for a given subobject). We can paste two pullback squares to obtain a bigger pullback square (rectangle):

$$\begin{array}{ccccc} S' & \longrightarrow & S & \longrightarrow & 1 \\ {\scriptstyle m'}\big\downarrow & & {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle \top} \\ Y & \xrightarrow{\quad f \quad} & X & \xrightarrow{\quad \phi_S \quad} & \Omega \end{array}$$

By the unicity clause in the definition of the subobject classifier, we must have $\phi_{S'} = \phi_S \circ f$. But this equation means exactly that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Sub}(X) & \xrightarrow{\quad \theta_X \quad} & \mathrm{Hom}(X, \Omega) \\ {\scriptstyle \mathrm{Sub}(f)}\big\downarrow & & \big\downarrow{\scriptstyle f^*} \\ \mathrm{Sub}(Y) & \xrightarrow{\quad \theta_Y \quad} & \mathrm{Hom}(Y, \Omega) \end{array}$$

Hence the isomorphism θ is natural as claimed, so Sub is a representable functor.

In light of the previous discussion, the obvious, reciprocal, question is: for a category to have a subobject classifier, is it enough for the subobject functor to be representable? The fact that this is true is Proposition 1 at page 33 of SGL:

> **Proposition.** A locally small category with all finite limits has a subobject classifier if and only if the subobject functor is representable. When that is the case, the category is well-powered.

Proof. The discussion in previous paragraphs shows the necessity of the representability of the subobject functor. To show its sufficience, suppose there exists a representation $\mathrm{Sub}(-) \cong \mathrm{Hom}(-, \Omega)$ for some representative object Ω. We need to show there exists a subobject classifier. Let Ω_0 be the "universal element" for the representation, that is, Ω_0 is the subobject corresponding to the identity arrow 1_Ω. By the Yoneda lemma, any arrow $\phi : X \to \Omega$ corresponds to the subobject $\mathrm{Sub}(\phi)(\Omega_0)$. Therefore, for any subobject S of X, there exists a unique arrow $\phi_S : X \to \Omega$ such that $S = \mathrm{Sub}(\phi_S)(\Omega_0)$. Because the action of the subobject functor on an arrow ϕ_S is to pullback Ω_0 along it, the arrow ϕ_S is the unique arrow making the following a cartesian square:

$$\begin{array}{ccc} S & \longrightarrow & \Omega_0 \\ {\scriptstyle \mathrm{Sub}(\phi_S)}\big\downarrow & & \big\downarrow \\ X & \xrightarrow{\phi_S} & \Omega \end{array}$$

We are almost done. For $\Omega_0 \rightarrowtail \Omega$ to be a subobject classifier, it is not enough that any monic arrow $m : S \rightarrowtail X$ is the pullback of a unique characteristic function, as is the case here. Additionally, we need Ω_0 to be the terminal object in the category: there needs to be exactly one arrow from any object X into Ω_0. The pullback square associated with 1_X (i.e. X seen as a subobject of itself) gives us an arrow $X \to \Omega_0$ for any object X, so we know there's always at least one. Suppose we have two arrows $\alpha, \beta : X \to \Omega_0$. Then the two following squares are pullback squares:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & \Omega_0 \\ {\scriptstyle 1_X}\big\downarrow & & \big\downarrow{\scriptstyle t_0} \\ X & \xrightarrow{t_0\alpha} & \Omega \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\beta} & \Omega_0 \\ {\scriptstyle 1_X}\big\downarrow & & \big\downarrow{\scriptstyle t_0} \\ X & \xrightarrow{t_0\beta} & \Omega \end{array}$$

Therefore $X = \mathrm{Sub}(t_0\alpha)(\Omega_0) = \mathrm{Sub}(t_0\beta)(\Omega_0)$, which implies $t_0\alpha = t_0\beta$ by unicity. Since t_0 is a monic arrow, this yields $\alpha = \beta$. Consequently there is at most one arrow from any object to Ω_0. Since we've already shown there's at least one, this means Ω_0 is a (the) terminal object, hence $\Omega_0 \to \Omega$ is a subobject classifier. ■

As is true for any representation, there is an isomorphism ι between any two representatives Ω and Ω', and it is the unique isomorphism which commutes with the representations. More precisely, pre-composition with ι yields a commutative diagram of functors and natural isomorphisms

$$\begin{array}{ccc} & & \mathrm{Hom}(-, \Omega) \\ & \overset{\cong}{\nearrow} & \big\downarrow{\scriptstyle \cong} \\ \mathrm{Sub}(-) & & \\ & \underset{\cong}{\searrow} & \\ & & \mathrm{Hom}(-, \Omega') \end{array}$$

Let Ω_0 and Ω_0' be universal elements for representations by Ω and Ω', respectively. Then, following both paths where the element $\Omega_0 \in \mathrm{Sub}(\Omega)$ goes yields $\Omega_0 =$

$\mathrm{Sub}(\iota)(\Omega_0')$. This means we have a pullback square

$$\begin{array}{ccc} \Omega_0 & \longrightarrow & \Omega_0' \\ {\scriptstyle \mathrm{Sub}(\iota)}\big\downarrow & & \big\downarrow \\ \Omega & \xrightarrow[\cong]{\iota} & \Omega' \end{array}$$

In the proof above, we saw that Ω_0 and Ω_0' are actually both the terminal object in our ambient category. Hence the top arrow in the previous diagram is an isomorphism. Therefore, a subobject classifier is unique up to (unique) isomorphism. From now on, we will say *the* subobject classifier, when it exists.