On integral morphisms

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Let $\phi : B \to A$ be an algebra. We say that an element $a \in A$ is **integral** over B if there exists some monic polynomial $p \in B[x]$ such that $p(a) = 0$ in A. This is a generalization of the concept of *algebraic elements* in the theory of fields. The structure morphism ϕ is said to be an **integral morphism** if every element in A is integral over B; if ϕ is an injection, it is said to be an **integral extension**.

> **Proposition.** Let $\phi : B \to A$ be an algebra. If there exists elements $b_1, b_2, \dots, b_n \in B$ such that $B = (b_1, b_2, \dots, b_n)$, and if ϕ induces integral morphisms $B_{b_i} \to A_{\phi(b_i)}$ for every $1 \le i \le n$, then ϕ is integral.

Proof. To simplify matters, we start by supposing that ϕ is an inclusion and that B is a subring of A. Pick any $a \in A$; we want to show there exists a monic polynomial with coefficients in B whose roots include a. By hypothesis, for any $1 \le i \le n$, there exists a polynomial $p_i(x)$ with

$$p_i(x) \in B_{b_i} \cdot \{x^{m_i - 1}, x^{m_i - 2}, \dots, x^1, 1\}$$

such that $p_i(a) = a^{m_i}$. We can multiply each of these equations by the appropriate number of b_i's until we have "cleared the denominators", which produces new equations in A, so that

$$b_i^{t_i} a^{m_i} \in B \cdot \{a^{m_i - 1}, a^{m_i - 2}, \dots, a^1, 1\}.$$

By setting $t = \max\{t_1, t_2, \dots, t_n\}$, we have

$$b_i^t a^{m_i} \in B \cdot \{a^{m_i - 1}, a^{m_i - 2}, \dots, a^1, 1\}.$$

We can also set $m = \max\{m_1, m_2, \dots, m_n\}$ and multiply each $b_i^t a^{m_i}$ by enough copies of a to get it to be of the form $b_i^t a^m$, each multiplication by a increasing the "degree" of the terms in its expression as a "polynomial in a" by one, so we finally find

$$b_i^t a^m \in B \cdot \{a^{m-1}, a^{m-2}, \dots, a^1, 1\}.$$

Since $(b_1, b_2, \dots, b_n) = B$, it is true that $(b_1^t, b_2^t, \dots, b_n^t)$ generate the whole of B as well (hint: what is the form of a general element of any power of $(b_1, b_2, \dots, b_n)$?) Hence there exists elements $c_i \in B$ such that $c_1 b_1^t + c_2 b_2^t + \cdots + c_n b_n^t = 1$. Then

$$a^m = (c_1 b_1^t + c_2 b_2^t + \cdots + c_n b_n^t) a^m$$

so we find $a^m \in B \cdot \{a^{m-1}, a^{m-2}, \ldots, a^1, 1\}$. This proves a is the root of a polynomial with coefficients in B, i.e. a is integral over B. Since the element a was arbitrary, we have shown ϕ is integral in the case it is the inclusion of a subring into A. In the general case, we have that ϕ is integral if and only if the induced inclusion $\phi(B) \to A$ is integral. Using this, together with the fact that the morphism $B_{b_i} \to A_{\phi(b_i)}$ factorizes through $\phi(B)_{\phi(b_i)}$, one can reduce the general case to the one we just proved. ■

At the end of the previous proof, we used a special case of the following proposition in order to show that the general case held by reducing it to the special case of subring inclusion.

> **Proposition.** Consider the following commutative diagram in the category of commutative rings with unity $\mathbf{CRing}$:
>
> $$\begin{tikzcd} B \arrow[d, "\psi"'] \arrow[dr, "\phi"] & \\ B' \arrow[r, "\phi'"'] & A \end{tikzcd}$$
>
> If the top arrow ϕ is integral, then so is the bottom arrow ϕ'. Moreover, if ψ is surjective, then the converse of the previous statement is true as well. Note that injectivity of a morphism verifies the same kind of statement, so if the top arrow ϕ is an integral extension, then so is the bottom arrow ϕ'.

Because the proof is really straightforward, I decided to omit it. But the gist of it is: given any $a \in A$, there's a polynomial that kills it, and its coefficients look like $\phi(b)$ for some $b \in B$; therefore commutativity gives coefficients that look like $\phi'(\psi(b))$, and reciprocally if ψ is surjective. ■

Some immediate consequences of the previous proposition: let $\phi : B \to A$ be a ring homomorphism; then

- (quotient of B) for J any ideal of B contained in $\ker \phi$, the induced morphism $B/J \to A$ is integral if and only if ϕ is;
- (localization of B) for T any multiplicative subset of B, the induced morphism $T^{-1}B \to A$ is integral when ϕ is.

Things are also well-behaved when taking quotients of A. More precisely, let I be any ideal of A, and suppose $B \to A$ is an integral morphism. Then $B \to A/I$ is also integral. Indeed, pick any element $\overline{a} \in A/I$; because ϕ is integral, there exists an integer $n \geq 0$ and elements $b_i \in B$ such that we have

$$a^n + \phi(b_{n-1})a^{n-1} + \cdots + \phi(b_1)a^1 + \phi(b_0) = 0.$$

Hence simply passing this equation to the quotient gives us an expression for zero as a sum of powers of $\overline{a}$, with coefficients $\overline{\phi(b_i)}$. This shows the composite ring homomorphism $B \to A/I$ is integral, as we wanted.

Things are less well-behaved when localizing on A. More precisely, there exists a multiplicative subset $S \subseteq A$ and an integral morphism $B \to A$ such that the composite morphism $B \to S^{-1}A$ is not integral. Here's an example: choose $A = B = \kappa[t]$ the polynomials with coefficients in a field κ, and take the identity as the (obviously integral) morphism $B \to A$. Now I will show that $\kappa[t]_{(t)}$ is not integral over $\kappa[t]$. To do this, I need to exhibit some element that is not integral. After testing things out, I found that $1/(1-t)$ is probably the simplest element which is not integral over $\kappa[t]$. Here's a proof by contradiction: suppose it *is* integral, so there exists an integer $n \geq 0$ and polynomials $p_i(t) \in \kappa[t]$ such that

$$\left(\frac{1}{1-t}\right)^n + p_{n-1}(t)\left(\frac{1}{1-t}\right)^{n-1} + \cdots + p_1(t)\frac{1}{1-t} + p_0(t) = 0.$$

Putting everything over a common denominator and rewriting, we obtain

$$\frac{1 + (1-t)p_{n-1}(t) + \cdots + (1-t)^{n-1}p_1(t) + (1-t)^n p_0(t)}{(1-t)^n} = 0.$$

Because $\kappa[t]$ is an integral domain, this equality in $\kappa[t]_{(t)}$ implies the following equality in $\kappa[t]$:

$$1 + (1-t)p_{n-1}(t) + \cdots + (1-t)^{n-1}p_1(t) + (1-t)^n p_0(t) = 0.$$

Evaluating the left-hand side at $t = 1$ yields $1 = 0$, which is absurd because $\kappa[t]$ is not the trivial ring. Therefore the rational expression $1/(1-t)$ cannot be an integral element of $\kappa[t]_{(t)}$ over $\kappa[t]$, whence the ring homomorphism $\kappa[t] \to \kappa[t]_{(t)}$ is not integral.

It is still possible to obtain an integral morphism by localization at A and B *at the same time*, in some cases: if $\phi : B \to A$ is an integral morphism and if $T \subseteq B$ is a multiplicative subset of B, then the induced morphism $T^{-1}B \to \phi(T)^{-1}A$ is also an integral morphism. The proof is as follows. Pick any $a/\phi(t)$ in the localized ring $\phi(T)^{-1}A$. As usual, the fact ϕ is integral means there exists an integer $n \geq 0$ and elements $b_i \in B$ such that

$$a^n + \phi(b_{n-1})a^{n-1} + \cdots + \phi(b_1)a + \phi(b_0) = 0.$$

Denote by $\widetilde{\phi}$ the induced morphism $T^{-1}B \to \phi(T)^{-1}A$. Passing the previous equation to the localization and dividing out by $\phi(t)^n$ gives an equation

$$\left(\frac{a}{\phi(t)}\right)^n + \widetilde{\phi}\left(\frac{b_{n-1}}{t}\right)\left(\frac{a}{\phi(t)}\right)^{n-1} + \cdots + \widetilde{\phi}\left(\frac{b_1}{t^{n-1}}\right)\frac{a}{\phi(t)} + \widetilde{\phi}\left(\frac{b_0}{t^n}\right) = 0.$$

This shows $\widetilde{\phi}$ is an integral morphism, as we wanted. ■

> **Proposition.** Let $\phi : B \to A$ be an algebra. An element $a \in A$ is integral if and only if it is contained in a finite subalgebra of A. Consequently, if A itself is a finite algebra, then ϕ is an integral morphism.

Proof. If a is integral, then it satisfies a monic polynomial of degree n with coefficients in B. Hence $B \cdot \{a^{n-1}, \dots, a, 1\}$ is a B-submodule of finite type which is closed under multiplication (i.e. a finite subalgebra) and which contains a. On the other hand, suppose a is contained in a finite subalgebra B generated by elements $m_1, m_2, \dots, m_n \in B$. For each $1 \le i \le n$, write am_i as the linear combination $\sum_{j=1}^n b_{ij} m_j$, so we have the matrix equation

$$\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix} = a \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix}.$$

We write C for the square $n \times n$ matrix, and m for the vector. Then the previous equation may be written equivalently as $(aI - C)m = 0$. We would like to conclude that the determinant $\det(aI - C)$ is zero, but we can't do that in general: a matrix is invertible (in a general ring) if and only if its determinant is invertible. Thus we know the determinant is not invertible, but we can't go further without the following trick. Recall that any square matrix M has an *adjugate matrix* $\operatorname{adj}(M)$ such that $\operatorname{adj}(M)M = \det(M)I$. In our case, multiplying our equation on both sides by $\operatorname{adj}(aI - C)$ yields

$$\det(aI - C)m = 0.$$

The determinant $\det(aI - C)$ is an element of B that kills every generator m_i of B. Therefore, it kills every element in B; in particular, it kills $1 \in B$ so we get $\det(aI - C) = 0$. Expanding out the determinant yields an integral expression for a with coefficients in B. ■

The previous result is quite practical. For instance, as a sort of complement of the earlier proposition with the commutative diagram, we use it to show integral morphisms behave well under composition:

> **Proposition.** Let $\phi : B \to A$ and $\psi : C \to B$ be two integral morphisms. Then their composition $\phi \circ \psi$ is also an integral morphism.

Proof. Suppose first that ϕ and ψ are actually inclusions of rings, so that $C \subseteq B \subseteq A$. Pick any element $a \in A$. Because A is integral over B, there exists an integer $n \ge 0$ and elements $b_i \in B$ such that

$$a^n = b_{n-1}a^{n-1} + \dots + b_1 a + b_0.$$

Likewise, since each b_i is integral over C, one can choose a large enough integer $m \geq 0$ such that, for all $0 \leq i \leq n-1$, the power b_i^m can be expressed as a C-linear combination of strictly smaller powers of b_i. Now consider the C-submodule A' of A defined as

$$A' = C \cdot \{a^k b_1^{e_1} b_2^{e_2} \cdots b_n^{e_n} \mid 0 \leq k \leq n-1, 0 \leq e_i \leq m-1\}.$$

Now I claim A' is actually a C-subalgebra, i.e. it is closed under multiplication. This fact can be verified by showing that the product of any two generators is again an element of A'. It is clear that the product of any two generators has the form $a^k b_1^{e_1} b_2^{e_2} \cdots b_n^{e_n}$ since we work in commutative rings, but the exponents could be “too large”. However, this is not a problem because of the following algorithm. If k is larger than $n-1$, we can rewrite a^k as a B-linear combination of powers of a that are small enough, and each coefficient of that B-linear combination is a product of the b_i’s. After distributing everything, we get a new C-linear combination, where each monomial has power of a not larger than $n-1$. If, after this operation, a monomial has a power of b_i larger than $m-1$, we apply the same idea to write it as a C-linear combinations of smaller powers of b_i. This roughly shows that A' is a C-subalgebra, and it is clearly finite because it’s defined using a finite number of generators. Since it contains a, the previous proposition ensures a is integral over C. Because a was an arbitrary element of A, the claim is proven in the case ϕ and ψ are ring inclusions. The general case uses the same idea, so I won’t bother writing it down. ■