On algebras over a ring

written by rapha on Functor Network
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In this post, all rings are commutative with unit.

Let A and B be two rings. We say that A is a **B-algebra**, or an **algebra over B**, if A is also a B-module, in such a way that the ring addition is the same as the module addition, and scalar multiplication satisfies

$$b \cdot (xy) = (b \cdot x)y = x(b \cdot y).$$

A **morphism** of B-algebras is a B-linear ring homomorphism. Explicitely:

- $\phi(x + y) = \phi(x) + \phi(y)$;
- $\phi(xy) = \phi(x)\phi(y)$;
- $\phi(b \cdot x) = b \cdot \phi(x)$; and
- $\phi(1) = 1$.

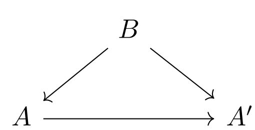
The B-algebras and their morphisms form a category, denoted **B-Alg**.

Notice that given $b \in B$, we can produce $b \cdot 1 \in A$. This gives us a quite "natural" function $B \to A$. This function is a ring homomorphism because it sends $1 \in B$ to $1 \in A$ and it respects addition by the module axioms, and it respects multiplication by the above axiom:

$$(b \cdot 1)(b' \cdot 1) = b \cdot (1(b' \cdot 1)) = b \cdot (b' \cdot 1) = (bb') \cdot 1.$$

On the other hand, given a ring homomorphism $B \to A$, we may use it to define scalar multiplication in terms of the multiplication in A, and this gives a B-algebra structure to A. Hence the data of a B-algebra structure on A is equivalent, in a sense we'll make precise, to a morphism of rings $B \to A$.

Fixing a ring B, there's the **category of objects under** B, denoted by $B \downarrow$ **CRing**, which is a special case of the comma category construction where the objects are ring homomorphisms $B \to A$ where A ranges over the objects of **CRing**, and where the arrows are ring homomorphisms $A \to A'$ such that the following diagram commutes:



Our claim is that there's an *isomorphism* of categories (this is stronger than equivalence) between the category **B-Alg** and the category of objects under B. The proof is annoying so we omit it; constructing the correct functors that make up the isomorphism is quite easy. The only interesting part is: given an object $\phi : B \to A$ in $B \downarrow \mathbf{CRing}$, we can define a B-algebra structure on A by specifying $(b, x) \mapsto \phi(b)x$ as the action of B on A, and given some B-algebra A, we can construct a ring homomorphism $B \to A$ by sending b to $b \cdot 1$, just as we did earlier.

From this result, we can say that a B-algebra structure on A is precisely a ring homomorphism $B \to A$, which we call the **structure morphism**; now a B-algebra homomorphism is a ring homomorphism that commutes with the structure morphisms.

Here is some more terminology. Let $B \to A$ be an algebra.

- We say A is an **algebra of finite type** (French: *algèbre de type fini*) when there exists a finite set of elements of A that are able to generate A using the three available operations. In other words, an algebra is of finite type if and only if there is a surjective algebra homomorphism

  $$B[x_1, x_2, \ldots, x_n] \to A$$

  which sends each variable to a generator.

- We say A is a **finite algebra** if it is of finite type *as a B-module*, that is, when there exists a finite set of elements of A that are able to generate A using only addition and scalar multiplication. Hence, an algebra is finite if and only if there is a surjective B-module homomorphism

  $$B^{\oplus n} \to A$$

  sending the unit in each copy of B to a generator.

An algebra A over a field κ is finite if and only if A is a finite-dimensional vector space over κ, which explains the terminology a little bit.