On the cotangent space on a smooth manifold, defined from its structure sheaf

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Any smooth manifold M comes with its sheaf of rings of smooth functions. This sheaf labels each open set U in M with the set $\mathcal{C}^\infty(U)$ of smooth, real-valued function defined over U. Given any point $p \in M$, the **stalk** at p, written $\mathcal{C}^\infty_p$, is defined to be the direct limit over the directed set of all open neighborhoods of p (this directed set is written as $\mathcal{N}_p$):

$$\mathcal{C}^\infty_p = \varinjlim_{U \in \mathcal{N}_p} \mathcal{C}^\infty(U).$$

Elements of $\mathcal{C}^\infty_p$ are called germs at p and they are essentially "shreds" of a smooth function at a point. They can be interpreted somewhat abstractly as the set of functions that are defined "close" to p, on an infinitesimal (or, if you prefer, arbitrary small) open neighborhood around p. Concretely, germs are equivalence classes, each one represented by a smooth function f defined on some open set containing p. For any germ $[f]$ at p, we may evaluate any representative function f at p, and this gives a well-defined morphism of rings, called the **evaluation** at p:

$$\mathrm{ev}_p : \mathcal{C}^\infty_p \to \mathbb{R}.$$

The kernel of ev_p is precisely the ideal of smooth functions which vanish at p, which we denote I_p. Because $\mathcal{C}^\infty_p / I_p \cong \mathbb{R}$, the ideal I_p is actually a maximal ideal. Since any germ not in I_p is represented by a smooth function which is nonzero and thus invertible around p, the ideal I_p is the *only* maximal ideal of the ring $\mathcal{C}^\infty_p$. We say in that case $\mathcal{C}^\infty_p$ is a **local ring**, and M is a **locally ringed space** because $\mathcal{C}^\infty_p$ is a local ring at every point of the smooth manifold M.

Recall how products of ideals work: the ideal I_p^2 is the ideal generated by products of the form $[f][g]$, with both $[f]$ and $[g]$ germs at p that vanish there. Because $I_p^2 \subseteq I_p$, we can take the quotient I_p/I_p^2. This ideal in $\mathcal{C}^\infty_p$ is actually a real vector space, in a natural way (hint: $\mathcal{C}^\infty_p / I_p \cong \mathbb{R}$). It can be used as the definition of the **cotangent space** at p on the manifold, written as $T^*_p M$. If M is of dimension n then, by the multivariate version of Taylor's theorem, any smooth function can be written, in local coordinates $x = (x^1, x^2, \ldots, x^n)$ around p, as

$$f(x) = D(x) + \sum_{|\alpha|=2} (c_\alpha + h_\alpha(x)) x^\alpha$$

where D is a linear map $\mathbb{R}^n \to \mathbb{R}$, each α is a multi-index as usual, each c_α is some real number, and each h_α is a function such that $\lim_{x\to 0} h_\alpha(x) = 0$. From this, we see that the germ $[f]$ is represented by $[D]$ in $T^*_p M$, since the part where we sum over indices α with $|\alpha| = 2$ is killed when we quotient out I_p^2. Now we see that T_p^*M is actually a finite real vector space of dimension n, with basis the set of (classes of) germs represented by the linear maps which are defined around p and which send, in the local coordinates, a point to its i-th coordinate. We write these linear maps as dx_i. Now the **tangent space** at a point p on a smooth manifold M, written as T_pM, can be defined as the dual of the vector space T_p^*M.

The construction of the tangent space from the cotangent space is interesting. It shows how the cotangent space is somehow more algebraically natural, while the tangent space is obviously more natural from a geometrical point of view. This illustrate a general phenomenon, where geometry and algebra are two sides of the same coin. But this is also interesting because we only used the locally ringed space structure on the manifold M (the fact we have smooth functions was only used to show it corresponds to the usual definition). Hence we can go through the cotangent space construction in order to build tangent spaces at points on any locally ringed space, such as schemes.