Skyscraper sheaves

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Let X be a topological space, let p be a point of X, and let S be any set. In the notation of Vakil, define the **skyscraper sheaf** supported at p by the formula

$$(i_{p,*}S)(U) = \begin{cases} S & \text{if } p \in U; \\ 1 & \text{otherwise.} \end{cases}$$

Here 1 is the singleton set. We can also define such a sheaf in other categories (abelian groups, rings, etc.), replacing the singleton set by the appropriate terminal object.

If $V \subseteq U$ is a containement of open sets, then the restriction $\rho_{U,V}$ is given as follows:

- if $p \in V$, then $\rho_{U,V}$ is the identity on S;
- otherwise, $\rho_{U,V}$ is the unique map to 1.

For $W \subseteq V \subseteq U$, we have $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$, because if $p \notin W$ both sides are the unique map to 1, and otherwise both sides are the identity map on S. Therefore we have a presheaf of sets on X (and this could also be a presheaf of other objects as well, as long as there is a terminal object).

Suppose $\{U_i\}_{i \in I}$ is an open cover of some open set U in X, and suppose $\{s_i \in (i_{p,*}S)(U_i)\}_{i \in I}$ is a collection of sections such that, for any $i, j \in I$, we have $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. If $p \notin U$, then the unique element of $(i_{p,*}S)(U)$ is evidently the unique gluing of the sections. If $p \in U$, then there is some $i_0 \in I$ such that $p \in U_i$; let $s = s_{i_0}$ be the gluing of the sections. Because restriction to U_{i_0} is the identity, the gluing is clearly unique. We want to show that for any $i \in I$, we have $s|_{U_i} = s_i$. If $p \notin U_i$, then it's obviously true. If $p \in U_i$, then $p \in U_i \cap U_{i_0}$, and since $s_i|_{U_i \cap U_{i_0}} = s_{i_0}|_{U_i \cap U_{i_0}}$ with these restrictions being the identity, we find $s_i = s_{i_0} = s = s|_{U_i}$, as required. This was a lot of words to say a simple thing: we have a sheaf.

The "skyscraper" in the name is explained by the following fact:

> The stalk of $i_{p,*}S$ at a point $q \in X$ is S if q is in the closure of p, and is the singleton set 1 otherwise.

That's not very hard to show. Suppose that q is not in the closure of p. Then there exists some open set around q that does not contain p. Hence any germ in the stalk at q can be represented by an element in the singleton set 1, which means all germs are equal and the stalk may be identified with 1. On the other hand, suppose that q is in the closure of p. This means all open sets which contain q *also* contain p. The stalk is a colimit, and now we're saying it's a colimit over a constant diagram (every object in the diagram is S). Therefore, the colimit is S.

Note that we can argue more abstractly for the first case, when q is not in the closure, in a way that shows the stalk is the terminal object 1 in other categories (abelian groups, rings, etc). The stalk is a direct limit which is computed over the directed set of all opens containing p, ordered by reverse inclusion. Recall that a directed set is a poset $(A, \leq)$ in which every pair of elements has an upper bound. A subset $(B, \leq)$ of a poset is said to be **cofinal** in A if, for every $a \in A$, it is possible to find some $b \in B$ such that $a \leq b$. For instance, when q is not contained in the closure of p, the set of open neighborhoods of q that do not contain p is cofinal in the directed set of all open neighborhoods of q, ordered by reverse inclusion. Note that any cofinal set in a directed set is also directed. One can show that the direct limit computed over a directed set is equal (or more precisely, isomorphic up to a unique canonical isomorphism) to the direct limit computed over the "smaller" cofinal set. In our example, this means the stalk at q is 1 in any category with such a terminal object, because the direct limit can be computed over the cofinal set of neighborhoods not containing p, and that's a constant diagram with all objects equal to 1.

From the previous discussion, skyscraper sheaves look like a skyscraper towering above a point, and this mental picture is accurate when the point p is closed. When a point is not closed (such a situation happens frequently in algebraic geometry), there are some points "nearby" over which the stalk is also S, so it looks like a city's downtown more than a single skyscraper.

What About the Weird Notation?

The notation $i_{p,*}S$ is weird, but it makes sense in light of the following construction. Let $f : X \to Y$ be a (continuous) map of topological spaces, and let $\mathscr{F}$ be a sheaf on X. We define the **pushforward** of $\mathscr{F}$ along f to be the sheaf defined by the equation

$$(f_*\mathscr{F})(U) = \mathscr{F}(f^*U),$$

where f^* denotes the inverse image (or preimage) of f (it's more often written as f^{-1} but I prefer the notation with a star). Because f is continuous, the inverse image of an open set is an open set, so the previous equation makes sense. Given $V \subseteq U$ an inclusion of open sets in Y, the restriction from U to V is defined by the equation

$$\rho^{f_*\mathscr{F}}_{U,V} = \rho^{\mathscr{F}}_{f^*U,f^*V}.$$

This defines a presheaf, simply because $\mathscr{F}$ itself is a presheaf: clearly the restriction from an open set to itself is the identity, and

$$\begin{aligned}\rho^{f_*\mathscr{F}}_{V,W} \circ \rho^{f_*\mathscr{F}}_{U,V} &= \rho^{\mathscr{F}}_{f^*V,f^*W} \circ \rho^{\mathscr{F}}_{f^*U,f^*V} \\ &= \rho^{\mathscr{F}}_{f^*U,f^*W} \\ &= \rho^{f_*\mathscr{F}}_{U,W}.\end{aligned}$$

The fact $\mathscr{F}$ is a sheaf is also sufficient to make its pushforward a sheaf as well. Suppose U is an open set in Y, and $\{U_i\}_{i\in I}$ is an open cover of U. For each $i \in I$, let $s_i \in f_*\mathscr{F}(U_i)$ and suppose further that, for any $i, j \in I$, we have $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. We want to show the existence of a unique section $s \in f_*\mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for each $i \in I$. Each section s_i is an element of $\mathscr{F}(f^*U_i)$, and the fact these sections all agree on overlaps $U_i \cap U_j$ together with the fact $f^*(U_i \cap U_j) = f^*U_i \cap f^*U_j$ means there exists a unique $s \in \mathscr{F}(f^*U)$ with the desired property. Notice that a key part of why the pushforward is a sheaf, is the fact the inverse image preserves both arbitrary unions and intersections (union is because we need the collection $\{f^*U_i\}_{i\in I}$ to be an open cover of f^*U).

To make sense of the notation for skyscraper sheaves, we also need to talk about the **constant sheaf**. Let S be any set. The constant sheaf associated to S, denoted $\underline{S}$, is defined by labeling each open set U with the set of functions $U \to S$ that are locally constant (i.e. around each point of U there exists some open set contained in U on which the function is constant – this is the same as requiring the function to be constant on connected components of U). Restriction is the usual restriction of maps, which obviously respects the presheaf condition. The sheaf axiom is not hard to check either.

Back to skyscrapers. Let $i_p : 1 \to X$ be the "inclusion map" which points to $p \in X$. We consider $\underline{S}$ as a sheaf over the topological space 1. Let U be an open set in X. If $p \in U$, then i_p^*U is the unique point of 1, while on the other hand if $p \notin U$ then i_p^*U is the empty set. Hence we see the pushforward $i_{p,*}\underline{S}$ of the constant sheaf $\underline{S}$ is isomorphic in some obvious sense to the skyscraper sheaf $i_{p,*}S$ as defined earlier.