A simpler description for the ideal generated by the symmetric polynomials in two variables

written by rapha on Functor Network
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The goal of this short post is to convince myself that the ideal generated by Sym^+ inside of $\mathbb{C}[x,y]$ can be more simply described as the ideal generated by xy and $x+y$, i.e.

$$\langle \mathrm{Sym}^+ \rangle = \langle xy, x+y \rangle.$$

As a shorthand, set $A = \mathbb{C}[x,y]$. The ring A is graded:

$$A = \bigoplus_{d=0}^{\infty} A^{(d)},$$

where $A^{(d)}$ is the $\mathbb{C}$-module consisting of the homogeneous polynomials of degree d:

$$A^{(d)} = \mathbb{C}\{x^a y^b \mid a+b=d\}.$$

In general, a polynomial is said to be *symmetric* when it is invariant under any permutation of the variables. In our case, a polynomial $p(x,y) \in A$ is symmetric when $p(x,y) = p(y,x)$. For instance, x^3+y^3+2xy is symmetric while $x+y^2$ is not. The product and difference of two symmetric polynomials is also a symmetric polynomial. Also, 1 is a trivial example of a symmetric polynomial. Hence the set of all symmetric polynomials is a subring of A, which we denote by Sym. This subring is naturally graded:

$$\mathrm{Sym} = \bigoplus_{d=0}^{\infty} \mathrm{Sym}^{(d)}$$

where $\mathrm{Sym}^{(d)} = \mathrm{Sym} \cap A^{(d)}$ is the set of symmetric homogeneous polynomials of degree d.

Now we restrict our attention to the set of symmetric polynomials which have 0 as a root. This is exactly the set

$$\mathrm{Sym}^+ = \bigoplus_{d=1}^{\infty} \mathrm{Sym}^{(d)}.$$

Let's prove that every element in Sym^+ can be written as an A-linear combination of xy and $x+y$, which will show the first equation between generated ideals at the top of this post holds. Concretely, every element $p(x,y)$ in Sym^+ can be written as some sum of homogeneous elements, all of degree at least one. If I can show that each of these homogeneous elements can be written in the form

$$\text{[some polynomial]} \cdot (x+y) + \text{[some other polynomial]} \cdot xy,$$

then simply by grouping together terms in $x+y$ and terms in xy and factoring out, I will obtain an expression for $p(x,y)$ as an A-linear combination of $x+y$ and xy. So we can reduce the problem to $p(x,y)$ being an homogeneous polynomial of degree $d \geq 1$. We can even do more. Recall that $\mathrm{Sym}^{(d)}$ is a vector space over $\mathbb{C}$, and if we can show every basis element can be written as a linear combination like we want, then we have shown $p(x,y)$ can be written like that as well. Hence we have reduced the problem to showing that for any $d \geq 1$, some basis of $\mathrm{Sym}^{(d)}$ can be written as an A-linear combination of $x+y$ and xy.

Let's chose the m-basis. I hope to write some post about this basis. However I want to keep this one short, so here are the basics for future quick recalling. Let's say that a *partition* (French: *partage*) of some natural number $n \geq 0$ is some list $\lambda = (\lambda_1, \dots, \lambda_k)$ of natural numbers with $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$ and such that $\lambda_1 + \cdots + \lambda_k = n$. The integer k is the numbers of *parts* of λ, which we may write as $\ell(\lambda)$. To indicate that λ is a partition of n, we write $\lambda \vdash n$.

Fix some degree $d \geq 1$. To build a basis for $\mathrm{Sym}^{(d)}$, pick some $\lambda \vdash d$ with $\ell(\lambda) = 2$ (if needed, extend the partition with a second part of zero length). For instance, if $d = 3$, then possible choices of partition are $(3,0)$ and $(2,1)$. Now write

$$m_\lambda = x^{\lambda_1} y^{\lambda_2} + x^{\lambda_2} y^{\lambda_1}.$$

Collecting these polynomials for all possible partitions of d gives you a basis for $\mathrm{Sym}^{(d)}$. Note that what I've just written is a special case of the more general construction using the Reynolds symmetrization operator when there are more than two variables. Anyways, in the $d = 3$ example, the basis is given as $\{x^3 + y^3, x^2 y + x y^2\}$.

Let's get back to our original problem. Recall: fixing a degree $d \geq 1$, we need to show that each m_λ can be written in the form $p(x,y)(x+y) + r(x,y)xy$ for some polynomials p and q that obviously depend on m_λ. Here's the argument, which is quite simple after all this yapping. Suppose λ has two non-zero parts (eg. $\lambda = (2,1)$). Whatever m_λ is, it is garanteed by construction that each monomial in m_λ is divisible by both x and y; hence we can factor xy out of each monomial to obtain

$$m_{(\lambda_1, \lambda_2)} = xy \cdot m_{(\lambda_1 - 1, \lambda_2 - 1)}.$$

For instance,

$$m_{(2,1)} = x^2 y + x y^2 = xy(x+y) = xy \cdot m_{(1,0)}.$$

What if λ has a single non-zero part? Then $m_\lambda = m_{(d)}$ looks like this:

$$m_{(d)} = x^d + y^d.$$

Newton to the rescue:

$$\begin{aligned} x^d + y^d &= (x+y)^d - \sum_{i=1}^{d-1} \binom{d}{i} x^{d-i} y^i \\ &= (x+y)^d - xy \sum_{i=1}^{d-1} \binom{d}{i} x^{d-i-1} y^{i-1}. \end{aligned}$$

So we win. The only subtelty here is, what if $d = 1$ so then inside of the sum we have a x^{-1} term? Well, if $d = 1$ then the sum goes from $i = 1$ to $i = 0$, which means by convention that it's the empty sum, which is zero. Anyways, the case $d = 1$ can be done separately (hint: it's trivial).