Localization of modules as a left adjoint

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original link: https://functor.network/user/2593/entry/1157

Let A be a commutative ring with unit and let S be a multiplicative submonoid of A. Localization at S is a functor

$$S^{-1} : \mathbf{Mod}_A \to \mathbf{Mod}_{S^{-1}A}.$$

Writing $\ell : A \to S^{-1}A$ for the canonical ring localization homomorphism, we can use it to put an A-module structure on any $S^{-1}A$-module in a functorial way. In other words, we have a functor (*restriction of scalars*)

$$\ell^* : \mathbf{Mod}_{S^{-1}A} \to \mathbf{Mod}_A.$$

Since $\mathbf{Mod}_{S^{-1}A}$ is isomorphic to the full subcategory of $\mathbf{Mod}_A$ consisting of modules in which the action of every $s \in S$ is invertible, we can think of ℓ^* as a forgetful functor!

Claim. Localization is left adjoint to restriction of scalars: $S^{-1} \dashv \ell^*$. Spelled out, this means there is a natural isomorphism of sets

$$\mathrm{Hom}_{S^{-1}A}(S^{-1}M, N) \cong \mathrm{Hom}_A(M, \ell^* N).$$

Thinking of ℓ^* as a forgetful functor, this shines new light on localization of modules as a sort of "best inverse" to forgetting that one can multiply by S-fractions, or more precisely some sort of free construction (recall that free constructions are left adjoints to forgetful functors). In other words, and as expected, the localization $S^{-1}M$ is the "smallest" or "free-est" module in which one can divide by elements of S.

Proof of the claim. Recall that we characterized localization via universal arrows: for any A-linear map $f : M \to \ell^* N$, there exists a unique $S^{-1}A$-linear map $\theta(f) : S^{-1}M \to N$ such that $\ell^*(\theta(f)) \circ u_M = f$, where $u_M : M \to \ell^*(S^{-1}M)$ is the universal arrow.

We define $\eta_{M,N} : \mathrm{Hom}_{S^{-1}A}(S^{-1}M, N) \to \mathrm{Hom}_A(M, \ell^* N)$ by sending some $S^{-1}A$-linear map $\phi : S^{-1}M \to N$ to $\ell^*(\phi) \circ u_M$. The previous paragraph tells us that $\eta_{M,N}$ is a bijection.

The last thing to check is naturality. It suffices to check that for any A-linear map $\alpha : M' \to M$, we have $\ell^*(S^{-1}\alpha) \circ u_{M'} = u_M \circ \alpha$. However, this is immediate because we *defined* $S^{-1}\alpha$ as the unique $S^{-1}A$-linear map such that this is verified! ■

Now recall that extension of scalar has a more "usual" left adjoint, called *extension of scalars*, denoted in our case by

$$\ell_! : \mathbf{Mod}_A \to \mathbf{Mod}_{S^{-1}A}.$$

Its action on objects is given by $\ell_!(M) = M \otimes_A S^{-1}A$. Because of the unicity of adjoints up to a unique canonical isomorphism, there is a natural isomorphism of functors

$$S^{-1}(-) \cong (-) \otimes_A S^{-1}A.$$

So localization enjoys the same universal property as this tensor product, and similarily this tensor product enjoys the same universal property as the localization.