Torsion modules

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This will be a very short post about torsion modules. I especially wanted to flesh out an easy characterization of them. I will add more examples eventually…

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Let A be any commutative ring.

Definition. An A-module M is said to be **torsion-free** if $a \cdot m = 0$ implies either that a is a zero divisor in A, or that $m = 0$.

[TODO Add some examples and non-examples.]

From this point, the ring A will always be an integral domain. In this case, the module M is torsion-free if and only if the equation $a \cdot m = 0$ implies either that $a = 0$ or $m = 0$. Thus we see torsion-freeness as an analog of integrality for rings.

Definition. Given that A is an integral domain, the *torsion submodule* of any A-module M is the set M_{tors} of elements of M that are annihilated by some non-zero element of A.

It is really a submodule: if m_1 is annihilated by a_1 and m_2 is annihilated by a_2, then $m_1 + m_2$ is annihilated by $a_1 a_2$ — which is not zero because A is an integral domain. For similar reasons, M_{tors} is stable under the action of A.

Definition. Given that A is an integral domain, an A-module M is said to be *torsion* if $M = M_{\text{tors}}$, that is, if every element of M is annihilated by some non-zero element of A.

A cheap example of a torsion module is the abelian group $\mathbb{Z}/(n)$ for any integer $n > 0$.

Characterization. The module M is torsion if and only if

$$M \otimes_A K(A) = 0.$$

Proof. Suppose first that M is torsion, and let $m \otimes x$ be an elementary tensor. Let $a \in A$ be a non-zero element such that $a \cdot m = 0$. Thus

$$m \otimes x = m \otimes \frac{ax}{a} = (a \cdot m) \otimes \frac{x}{a} = 0.$$

On the other hand, suppose $M \otimes_A K(A)$ is the zero module. We identify this tensor product with $M_{(0)}$, the localization of M at the prime ideal (0). Now any fraction m/d is equal to zero by hypothesis, so there exists some nonzero $a \in A$ such that $a \cdot m = 0$ in M. Thus m, which is an arbitrary element of M, lies in M_{tors}. This shows $M_{\text{tors}} = M$, i.e. M is torsion. ■

In fact, the same reasoning shows that M_{tors} is the kernel of the canonical map

$$M \to M \otimes_A K(A).$$

This gives an answer to a question I asked myself a while ago. When working with tensor products, if $m \otimes 1 = 0$, can we guarantee that $m = 0$? Well, in our case, that question translates to: is the canonical map $M \to M \otimes_A K(A)$ an injection? That's the case if and only if M is torsion-free.