An example of a sheaf of modules which is not quasicoherent

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This is from *FoAG* (Ravi Vakil), exercise 6.1.B.

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Let k be any field, and fix X to be $\operatorname{Spec} k[t]$, the affine line over k. Let $\mathscr{F}$ be the skyscraper sheaf supported at the origin of X with group $k(t)$.

Over any distinguished open set $D(f)$, which may or may not contains the origin, there is a canonical ring homomorphism

$$\mathscr{O}_X(D(f)) \cong k[t]_f \to \mathscr{F}(D(f)),$$

which is either an inclusion of rings (if $D(f)$ contains the origin), or the zero homomorphism (if $D(f)$ doesn't contain the origin). In either case, this homomorphism gives to $\mathscr{F}(D(f))$ the structure of an $\mathscr{O}_X(D(f))$-algebra, so in particular a module structure.

By definition of skyscraper sheaves, it's obvious that restriction is compatible with the module action, since restriction to an open set which contains the origin does nothing, otherwise it sends everything to zero. Thus $\mathscr{F}$ is a sheaf of $\mathscr{O}_X$-modules on the distinguished base, and this structure may be extended in a canonical way to all open sets on X.

We are tasked to show that $\mathscr{F}$ is *not* a quasicoherent sheaf. Recall that a sheaf is **quasicoherent** if, for all affine open sets $U \subseteq X$, there exists an $\mathscr{O}_X(U)$-module M and an isomorphism $\mathscr{F}|_U \cong \widetilde{M}$ of $\mathscr{O}_U$-modules.

We're going to prove this by way of contradiction: suppose $\mathscr{F}$ *is* quasicoherent. The whole space X is affine, so there must exist some $k[t]$-module M such that $\mathscr{F} \cong \widetilde{M}$. In particular, there is an isomorphism between the global sections, i.e. $M \cong k(t)$ as $k[t]$-modules. Hence we find that

$$\mathscr{F} \cong \widetilde{k(t)}.$$

Now, pick some open set that doesn't contain the origin (for instance, the distinguished open set $D(t)$). Because $\mathscr{F}$ is a skyscraper sheaf supported at the origin, $\mathscr{F}(D(t)) = 0$. By the above isomorphism, we find

$$\widetilde{k(t)}(D(t)) \cong k(t) \cong 0.$$

This is clearly a contradiction since at least t is not zero in $k(t)$. We have shown that $\mathscr{F}$ cannot be quasicoherent. ■

Let's spice things up a bit and define everything in the same way as before, except that now $\mathscr{F}$ is supported at the generic point η of X instead of at the origin. We want to prove, dear reader, that in this case $\mathscr{F}$ really *is* a quasicoherent sheaf.

This is not too difficult to see. First of all, it thankfully suffices to prove there exists a single $k[t]$-module M such that $\mathscr{F} \cong \widetilde{M}$ (that's because X is an affine scheme, plus Theorem 6.1.2 that being quasicoherent is affine-local). Now, every nonempty open set contains the generic point η, so every restriction map of $\mathscr{F}$ is the identity (except when we restrict to the empty set, in which case it is the zero map, of course). This is reflected by choosing M to be $k(t)$ (the global sections of $\mathscr{F}$), so that restriction to some distinguished open $D(f)$ changes nothing (f is already invertible in $k(t)$). Without going into all the trivial details, we connect these objects, proving $\mathscr{F}$ is quasicoherent:

$$\mathscr{F} \cong \widetilde{k(t)}.$$