Localization of modules as universal arrows

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Localization of commutative rings may be characterized by a unversal property whose morphisms are contained in the category of commutative rings. For modules, it is harder to formalize precisely since localizing changes the base ring, i.e. moves a module and its morphisms to a different category. It is the same phenomenon one encounters when dealing with the universal property characterizing free objects.

$* * *$

Given two functors $\mathbf{A} \xrightarrow{S} \mathbf{C} \xleftarrow{T} \mathbf{B}$, we can form the **comma category**, denoted $(S \downarrow T)$, as follows:

- objects are triples (a, b, h) where a is an object of $\mathbf{A}$, b is an object of $\mathbf{B}$, and h is an arrow $S(a) \to T(b)$.
- arrows from (a, b, h) to (a', b', h') are pairs (f, g) where f is an arrow $a \to a'$ and g is an arrow $b \to b'$, such that the following diagram commutes:

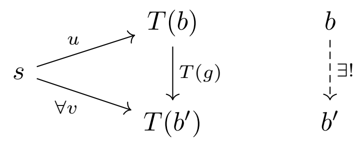
$$\begin{array}{ccc} S(a) & \xrightarrow{h} & T(b) \\ {\scriptstyle S(f)}\downarrow & & \downarrow{\scriptstyle T(g)} \\ S(a') & \xrightarrow[h']{} & T(b') \end{array}$$

Composition is defined by concatenation of commutative diagrams, which is clearly associative, and the identity arrow for the object (a, b, h) is simply $(\mathrm{id}_a, \mathrm{id}_b)$.

As a particular case, if S is a functor $* \to \mathbf{C}$, we identify S with the single object s to which it maps. An object of $(s \downarrow T)$ is called an *arrow from s to T*. A **universal arrow from s to T** is then simply an initial object in the category $(s \downarrow T)$.

Obviously, the data for an arrow from s to T may be compressed to a pair (b, h) instead of a triple. Similarily, the data for an arrow between (b, h) and (b', h') may be more simply given as a single arrow g making the appropriate diagram commute. Thus we can describe more explicitely a universal arrow from s to T:

- it is a pair (b, u) with b an object of $\mathbf{B}$ and u an arrow $s \to T(b)$
- such that for any other pair (b', v) with $v : s \to T(b')$
- there exists a unique $g : b \to b'$ in $\mathbf{B}$ such that $T(g) \circ u = v$



Because initial objects are unique up to a unique isomorphism, so are universal arrows.

We fix once and for all a commutative ring with unit A, together with a multiplicative submonoid $S \subseteq A$.

Let $T : \mathbf{Mod}_{S^{-1}A} \to \mathbf{Mod}_A$ be the functor defined by restriction of scalars along the canonical ring homomorphism $A \to S^{-1}A$. Now, given any A-module M, its **localization at** S is the universal arrow from M to T. We write this universal arrow as

$$M \to S^{-1}M$$

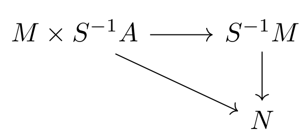
and often call it the "canonical localization morphism". By definition it is an A-linear map.

As with any characterization via a universal property, we get uniqueness for free but we need to work a bit to show existence. We will make an elementary construction that realizes the stated universal property. There are many ways to do it. We set

$$S^{-1}M := M \otimes_A S^{-1}A$$

and we define the A-linear map $\lambda : M \to S^{-1}M$ to be the obvious, canonical one given by the tensor product. The $S^{-1}A$-module structure on $S^{-1}M$ is given by $\frac{a}{s} \mapsto \mathrm{id} \otimes (\frac{a}{s} \cdot -)$.

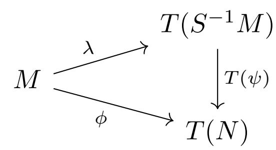
Now suppose N is some $S^{-1}A$-module, and that we have an A-linear map $\phi : M \to N$. Now, we may define a map $M \times S^{-1}A \to N$ given by $(m, a/s) \mapsto (a/s) \cdot \phi(m)$, which is obviously A-bilinear. Then the universal property of the tensor product produces a canonical A-linear map $\psi : S^{-1}M \to N$, the unique one such that the following diagram commutes:



Given $f \in S^{-1}A$ and an elementary tensor $m \otimes f' \in S^{-1}M$,

$$\begin{aligned} \psi(f \cdot (m \otimes f')) &= \psi(m \otimes ff') \\ &= (ff') \cdot \phi(m) \\ &= f \cdot (f' \cdot \phi(m)) \\ &= f \cdot \psi(m \otimes f') \end{aligned}$$

so that ψ is actually an $S^{-1}A$-linear map, and moreover this fact only depends on the previous diagram commuting and the $S^{-1}A$-module structure on $S^{-1}M$. We see that ψ is a morphism between $S^{-1}M$ and N in the category $\mathbf{Mod}_{S^{-1}A}$, and it makes the following diagram commute in $\mathbf{Mod}_A$:



Let's tackle unicity. Suppose α and β are two $S^{-1}A$-linear maps $S^{-1}M \to N$ such that we have $T(\alpha) \circ \lambda = T(\beta) \circ \lambda$. We will show that in that case $\alpha = \beta$. To achieve this goal, it suffices to show α and β agree on elementary tensors $m \otimes (a/s)$. This is easy:

$$\begin{aligned} \alpha(m \otimes (a/s)) &= (a/s) \cdot \alpha(m \otimes 1) \\ &= (a/s) \cdot \beta(m \otimes 1) \\ &= \beta(m \otimes (a/s)). \end{aligned}$$

The construction is complete. ■

We let $\mathbf{B}$ be the full subcategory of $\mathbf{Mod}_A$ whose objects are A-modules such that, for all $s \in S$, scalar multiplication by s is an automorphism (i.e. all $s \in S$ have "invertible action" in the module). We can actually exhibit an *isomorphism* of categories between $\mathbf{B}$ and $\mathbf{Mod}_{S^{-1}A}$. They really are the same thing, in a strong sense.

I won't do the details of the isomorphism, but one interesting question is, how to put an $S^{-1}A$-module structure on some N object of $\mathbf{B}$? The answer is: given some fraction a/s in $S^{-1}A$ and some element $n \in N$, there exists some $n_s \in N$ such that $s \cdot n_s = n$; we define

$$(a/s) \cdot n := a \cdot n_s.$$