

# An exercise about closed embeddings of schemes

rapha · 29 May 2025

This is a small exercise I'm doing and I wanted to write down my solution here, why not. The precise exercise reference is *FoAG*, Ravi Vakil, p.253, 9.1.A.

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According to Vakil (*FoAG*, p.253), a morphism of schemes  $f : X \rightarrow Y$  is a **closed embedding** if it is an affine morphism (i.e. the preimage of any affine set  $\text{Spec } B$  is an affine set  $\text{Spec } A$ ) such that the corresponding ring homomorphism  $B \rightarrow A$  is surjective (i.e.  $A \cong B/I$  for some ideal  $I$ , and  $B \rightarrow B/I$  is the projection map).

Now the goal is to show that a closed embedding is a homeomorphism on its image (that is, show that  $f$  is an injective map, with continuous inverse), and moreover this image is a closed set in  $Y$ .

We start with showing  $f(X)$  is closed. But before we get to that, a small technical lemma will help us here:

**Lemma.** In a topological space having  $\mathcal{B}$  as a basis, a set is closed if and only if its intersection with every  $B \in \mathcal{B}$  is closed in the subspace  $B$ .

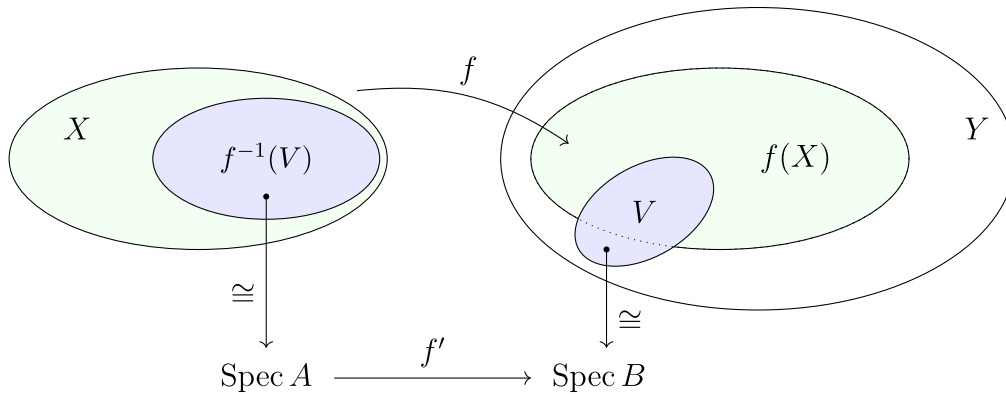
**Proof.** Suppose we have a subset  $A$  in the space such that, for all  $B \in \mathcal{B}$ , the set  $A \cap B$  is closed in the subspace  $B$ . Pick any point  $p$  in the closure of  $A$ ; we wish to show  $p$  lies in  $A$ . Because  $p$  is in the closure, there exists some basis element  $B \in \mathcal{B}$  such that  $p \in B$  and  $B \cap A \neq \emptyset$ . Now, let  $U$  be any open neighborhood of  $p$  in the subspace  $B$ . Because  $B$  is open,  $U$  is also open in the larger space. Then  $U \cap B$  is an open neighborhood of  $p$  in the larger space, so it must intersect  $A$ . Hence  $U$  intersects  $B \cap A$ . Since  $U$  was an arbitrary open neighborhood of  $p$  in the subspace  $B$ , this shows  $p$  lies in the closure of the closed set  $B \cap A$ , whence  $p \in B \cap A$ . In particular,  $p \in A$ , just as we wanted to show. ■

Now we do the exercise.

# Showing the image is closed.

Recall that the affine open sets form a basis for the topology on a scheme. Hence, by the lemma, it suffices to show that for any such affine open  $V$ , the intersection  $V \cap f(X)$  is closed in  $V$ .

Here's a picture of the argument to come:



In the picture,  $V$  is an affine open isomorphic to  $\text{Spec } B$ , and  $f^{-1}(V)$  is isomorphic to  $\text{Spec } A$ . The horizontal morphism  $f'$  between these spectra is obtained from  $f$  by composition of the vertical isomorphisms with the restriction of  $f$  to  $f^{-1}(V)$ . The morphism  $f'$  corresponds to a surjective homomorphism of rings  $B \rightarrow A$ , giving an isomorphism between  $A$  and  $B/I$  where  $I$  is the kernel. This means  $f'$  is a bijection between the prime ideals of  $A$  and the prime ideals of  $B$  which contain  $I$ , since the following diagram commutes by the construction of the isomorphism  $A \cong B/I$ :

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & \nearrow \cong & \\ B/I & & \end{array}$$

Rewriting the previous diagram in the category of affine schemes, we get

$$\begin{array}{ccc} \text{Spec } A & \xrightarrow{f'} & \text{Spec } B \\ \cong \downarrow & \nearrow & \\ \text{Spec } B/I & & \end{array}$$

This factorization shows that  $f'$  is a homeomorphism on its image, which is  $V(I) \subseteq \text{Spec } B$ . By the right hand vertical isomorphism, we get an homeomorphism between  $V(I)$  and the image of  $f$  restricted to  $f^{-1}(V)$ , which is  $f(f^{-1}(V))$ , which is  $V \cap f(X)$ . Therefore,  $V \cap f(X)$  is closed in the subspace  $V$ ; because  $V$  was an arbitrary basis element, the lemma implies  $f(X)$  is closed. ■

Parenthetically, this discussion shows that we may always suppose without loss of generality that the affine open  $f^{-1}(\text{Spec } B)$  is of the form  $\text{Spec } B/I$  when  $f$  is a closed embedding (and that the closed embedding itself restricts to a morphism of affine schemes  $\text{Spec } B/I \rightarrow \text{Spec } B$  which corresponds to the canonical projection  $B \rightarrow B/I$ .)

## Showing the map is injective.

The idea that shows the map is injective follows directly from what we've just discussed and unfolded about closed embeddings.

Suppose  $x_1$  and  $x_2$  are two points of  $X$  such that  $f(x_1) = f(x_2) = y$ . We want to show  $x_1 = x_2$ . Let  $V$  be an open affine neighborhood of  $y$ , and let  $U = f^{-1}(V)$  be its preimage by  $f$ . The previous discussion shows that  $f$  restricted to  $U$  is actually an homeomorphism onto  $V \cap f(X)$ . Clearly, we have  $y \in V \cap f(X)$ , and all possible preimages of  $y$ , including  $x_1$  and  $x_2$ , must lie in  $U$ . We see that for  $f$  to be a bijection when restricted to  $U$ , we must have the equality  $x_1 = x_2$ . This shows  $f$  is injective, since the two points with equal image were after all arbitrary. ■

## Takeaways

- If  $V$  is an open affine in  $Y$  and if  $U = f^{-1}(V)$ , then  $f|_U$  is a homeomorphism between  $U$  and  $V \cap f(X)$ .
- Writing  $V = \text{Spec } B$ , the set  $V \cap f(X)$  corresponds to prime ideals in  $B$  which contain  $I$ , where  $I$  is some ideal of  $B$  that is determined by  $f$  “locally”, that is, determined by what  $f|_U$  is as a map of affine schemes. More precisely, writing  $U = \text{Spec } A$ , the ideal  $I$  is the kernel of the corresponding surjective map  $B \rightarrow A$ .