An exercise about closed embeddings of schemes

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This is a small exercise I'm doing and I wanted to write down my solution here, why not. The precise exercise reference is *FoAG*, Ravi Vakil, p.253, 9.1.A.

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According to Vakil (*FoAG*, p.253), a morphism of schemes $f : X \to Y$ is a **closed embedding** if it is an affine morphism (i.e. the preimage of any affine set $\operatorname{Spec} B$ is an affine set $\operatorname{Spec} A$) such that the corresponding ring homomorphism $B \to A$ is surjective (i.e. $A \cong B/I$ for some ideal I, and $B \to B/I$ is the projection map).

Now the goal is to show that a closed embedding is a homeomorphism on its image (that is, show that f is an injective map, with continuous inverse), and moreover this image is a closed set in Y.

We start with showing $f(X)$ is closed. But before we get to that, a small technical lemma will help us here:

Lemma. In a topological space having $\mathcal{B}$ as a basis, a set is closed if and only if its intersection with every $B \in \mathcal{B}$ is closed in the subspace B.

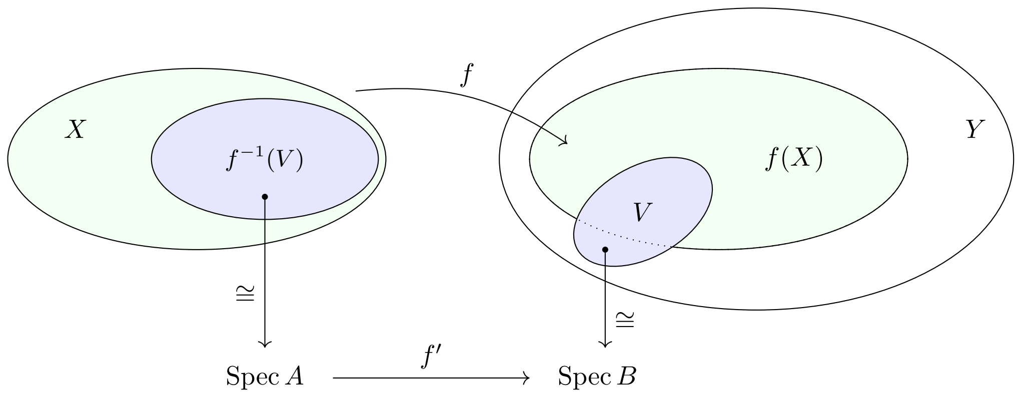
Proof. Suppose we have a subset A in the space such that, for all $B \in \mathcal{B}$, the set $A \cap B$ is closed in the subspace B. Pick any point p in the closure of A; we wish to show p lies in A. Because p is in the closure, there exists some basis element $B \in \mathcal{B}$ such that $p \in B$ and $B \cap A \neq \varnothing$. Now, let U be any open neighborhood of p in the subspace B. Because B is open, U is also open in the larger space. Then $U \cap B$ is an open neighborhood of p in the larger space, so it must intersect A. Hence U intersects $B \cap A$. Since U was an arbitrary open neighborhood of p in the subspace B, this shows p lies in the closure of the closed set $B \cap A$, whence $p \in B \cap A$. In particular, $p \in A$, just as we wanted to show. ■

Now we do the exercise.

Showing the image is closed.

Recall that the affine open sets form a basis for the topology on a scheme. Hence, by the lemma, it suffices to show that for any such affine open V, the intersection $V \cap f(X)$ is closed in V.

Here's a picture of the argument to come:



In the picture, V is an affine open isomorphic to $\operatorname{Spec} B$, and $f^{-1}(V)$ is isomorphic to $\operatorname{Spec} A$. The horizontal morphism f' between these spectra is obtained from f by composition of the vertical isomorphisms with the restriction of f to $f^{-1}(V)$. The morphism f' corresponds to a surjective homomorphism of rings $B \to A$, giving an isomorphism between A and B/I where I is the kernel. This means f' is a bijection between the prime ideals of A and the prime ideals of B which contain I, since the following diagram commutes by the construction of the isomorphism $A \cong B/I$:

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & \nearrow_{\cong} & \\ B/I & & \end{array}$$

Rewriting the previous diagram in the category of affines schemes, we get

$$\begin{array}{ccc} \operatorname{Spec} A & \xrightarrow{f'} & \operatorname{Spec} B \\ \cong\downarrow & \nearrow & \\ \operatorname{Spec} B/I & & \end{array}$$

This factorization shows that f' is a homeomorphism on its image, which is $V(I) \subseteq \operatorname{Spec} B$. By the right hand vertical isomorphism, we get an homeomorphism between $V(I)$ and the image of f restricted to $f^{-1}(V)$, which is $f(f^{-1}(V))$, which is $V \cap f(X)$. Therefore, $V \cap f(X)$ is closed in the subspace V; because V was an arbitrary basis element, the lemma implies $f(X)$ is closed. ■

Parenthetically, this discussion shows that we may always suppose without loss of generality that the affine open $f^{-1}(\operatorname{Spec} B)$ is of the form $\operatorname{Spec} B/I$ when f is a closed embedding (and that the closed embedding itself restricts to a morphism of affine schemes $\operatorname{Spec} B/I \to \operatorname{Spec} B$ which corresponds to the canonical projection $B \to B/I$.)

Showing the map is injective.

The idea that shows the map is injective follows directly from what we've just discussed and unfolded about closed embeddings.

Suppose x_1 and x_2 are two points of X such that $f(x_1) = f(x_2) = y$. We want to show $x_1 = x_2$. Let V be an open affine neighborhood of y, and let $U = f^{-1}(V)$ be its preimage by f. The previous discussion shows that f restricted to U is actually an homeomorphism onto $V \cap f(X)$. Clearly, we have $y \in V \cap f(X)$, and all possible preimages of y, including x_1 and x_2, must lie in U. We see that for f to be a bijection when restricted to U, we must have the equality $x_1 = x_2$. This shows f is injective, since the two points with equal image were after all arbitrary. ■

Takeaways

- If V is an open affine in Y and if $U = f^{-1}(V)$, then $f|_U$ is a homeomorphism between U and $V \cap f(X)$.
- Writing $V = \operatorname{Spec} B$, the set $V \cap f(X)$ corresponds to prime ideals in B which contain I, where I is some ideal of B that is determined by f "locally", that is, determined by what $f|_U$ is as a map of affine schemes. More precisely, writing $U = \operatorname{Spec} A$, the ideal I is the kernel of the corresponding surjective map $B \to A$.