The set of closed points is dense in many affine schemes

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In this post, we'll explore a particularly nice situation in which the set of closed points of an affine scheme is dense. Those affine schemes are closer to our geometric intuition than other types of schemes: that's because we expect points in a space to be "atomic", i.e. they make up everything, and so they are everywhere (they are "dense").

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Proposition. Let A be an algebra of finite type over a field k. In this situation, the closed points are a dense subset of $\operatorname{Spec} A$.

Proof. Because the distinguished open sets $D(f)$ form a basis for the topology on $\operatorname{Spec} A$, it suffices to show that any nonempty $D(f)$ contains a closed point.

Let f be an element of A that is not nilpotent. Because A is of finite type, the localization A_f is *also* of finite type (just add $1/f$ to some finite set of generators for A). Therefore, the canonical localization homomorphism $A \to A_f$ induces an inclusion map $\operatorname{Spec} A_f \to \operatorname{Spec} A$ which sends closed points to closed points. This inclusion map is an homeomorphism on its image, which is $D(f)$. Hence the image of any closed point in $\operatorname{Spec} A_f$ is a closed point in $D(f)$, and there exists at least one closed point in $\operatorname{Spec} A_f$ since A_f is not the zero ring (f is not nilpotent). ■

Here's a fun algebraic application of this schematic fact! We can use it to show that the algebra $k[x]_{(x)}$ is not finitely generated. By the proposition, it suffices to show that the set of closed points in $\operatorname{Spec} k[x]_{(x)}$ is not dense. This space is very simple: it only has two points, the generic point $\eta = (0)$ and the closed point $\mathfrak{p} = (x)$. But since $\mathfrak{p}$ is closed, that means η is an open point. In particular, there exists an open set, namely $\{\eta\}$, which contains no closed points. Therefore, the set of closed points, namely $\{\mathfrak{p}\}$, is not dense in the space, which shows $k[x]_{(x)}$ cannot be of finite type. Pretty neat!