

# The set of closed points is dense in many affine schemes

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In this post, we'll explore a particularly nice situation in which the set of closed points of an affine scheme is dense. Those affine schemes are closer to our geometric intuition than other types of schemes: that's because we expect points in a space to be "atomic", i.e. they make up everything, and so they are everywhere (they are "dense").

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**Proposition.** Let  $A$  be an algebra of finite type over a field  $k$ . In this situation, the closed points are a dense subset of  $\text{Spec } A$ .

**Proof.** Because the distinguished open sets  $D(f)$  form a basis for the topology on  $\text{Spec } A$ , it suffices to show that any nonempty  $D(f)$  contains a closed point.

Let  $f$  be an element of  $A$  that is not nilpotent. Because  $A$  is of finite type, the localization  $A_f$  is *also* of finite type (just add  $1/f$  to some finite set of generators for  $A$ ). Therefore, the canonical localization homomorphism  $A \rightarrow A_f$  induces an inclusion map  $\text{Spec } A_f \rightarrow \text{Spec } A$  which **sends closed points to closed points**. This inclusion map is an homeomorphism on its image, which is  $D(f)$ . Hence the image of any closed point in  $\text{Spec } A_f$  is a closed point in  $D(f)$ , and there exists at least one closed point in  $\text{Spec } A_f$  since  $A_f$  is not the zero ring ( $f$  is not nilpotent). ■

Here's a fun algebraic application of this schematic fact! We can use it to show that the algebra  $k[x]_{(x)}$  is not finitely generated. By the proposition, it suffices to show that the set of closed points in  $\text{Spec } k[x]_{(x)}$  is not dense. This space is very simple: it only has two points, the generic point  $\eta = (0)$  and the closed point  $\mathfrak{p} = (x)$ . But since  $\mathfrak{p}$  is closed, that means  $\eta$  is an open point. In particular, there exists an open set, namely  $\{\eta\}$ , which contains no closed points. Therefore, the set of closed points, namely  $\{\mathfrak{p}\}$ , is not dense in the space, which shows  $k[x]_{(x)}$  cannot be of finite type. Pretty neat!