# A situation in which the preimage of any maximal ideal stays maximal

rapha • 22 May 2025

For a general ring homomorphism, the preimage of a prime ideal stays prime. The situation is more complicated for maximal ideals: it's not true in general that a maximal ideal will stay maximal when pulled back! For a cheap example, look at the injection $\mathbb{Z} \to \mathbb{Q}$, for which the (maximal) ideal of $\mathbb{Q}$ gets pulled back to the *minimal* ideal of $\mathbb{Z}$. This technical advantage of prime ideals over maximal ideals was one of the driving forces behind scheme theory.

In this post, we will explore one particular situation where maximal ideals *are* pullback-friendly: when the rings are $k$-algebras, and the target is of finite type.

The notation $k$ will always denote some (any) field. Rings are commutative with unity.

* * *

Before we begin, we need a small technical lemma. Interesting in its own right, we will use this result to conclude that an integral domain is a field as soon as it admits a structure of a finite-dimensional vector space.

**Lemma.** If $A$ is an integral domain that is also a finite algebra over $k$, then $A$ is, in fact, a field.

**Proof.** Let $a$ be some non-zero element of $A$. The main idea is to show that the function $\lambda_a : x \mapsto ax$ is a $k$-linear automorphism. To do this, it suffices to show that $\lambda_a$ is an injection, because $A$ is a finite-dimensional vector space. But this is clear, since $A$ is an integral domain. ■

Parenthetically, notice that any integral domain admits a trivial vector space structure (multiplication by any scalar does nothing). Hence, if the integral domain has a finite number of elements, it is trivially a finite-dimensional vector space: by the lemma, it is a field!

Now we state the main point of this post:

**Proposition.** Let $\phi : A \to B$ be a morphism of algebras over $k$, and suppose further that $B$ is of finite type over $k$. Then the preimage by $\phi$ of any maximal ideal of $B$ is a maximal ideal of $A$.

In schematic terms, the induced morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ maps closed points to closed points, so it corresponds more closely to one's intuition of what a morphism of varieties "should" be!

**Proof.** Suppose $\mathfrak{m}$ is a maximal ideal of $B$. We want to show that the quotient ring $A/\phi^{-1}(\mathfrak{m})$ is a field. In general, the preimage of any maximal ideal is prime, so we already know $A/\phi^{-1}(\mathfrak{m})$ is an integral domain. We'll show that it is also a finite algebra over $k$, and then apply the lemma.

The morphism $\phi$ induces a canonical map $\widetilde{\phi} : A/\phi^{-1}(\mathfrak{m}) \to B/\mathfrak{m}$. It fits in the following commutative diagram of $k$-algebras and $k$-algebra homomorphisms:

$$\begin{array}{ccc} A & \xrightarrow{\quad\phi\quad} & B \\ \downarrow & & \downarrow \\ A/\phi^{-1}(\mathfrak{m}) & \xrightarrow{\quad\widetilde{\phi}\quad} & B/\mathfrak{m} \end{array}$$

Let's investigate the kernel of $\widetilde{\phi}$. Pick an arbitrary element $\overline{v}$ in $A/\phi^{-1}(\mathfrak{m})$ (where $v \in A$ and the overline denotes passing to the quotient, as usual). We have

$$\begin{aligned} \overline{v} \in \ker \widetilde{\phi} &\iff \widetilde{\phi}(\overline{v}) = 0 = \overline{\phi(v)} \\ &\iff \phi(v) \in \mathfrak{m} \\ &\iff v \in \phi^{-1}(\mathfrak{m}) \\ &\iff \overline{v} = 0. \end{aligned}$$

Hence the kernel is trivial, that is, $\widetilde{\phi}$ is an injective morphism.

As a last step, recall *Zariski's lemma* (sometimes also called the *Nullstellensatz*, although it seems to me like this term is overloaded!): if a field $K$ is of finite type as an algebra over some other field $k$, then $K$ is in fact finite over $k$, i.e. $\dim_k K < \infty$. In our case, applying $K = B/\mathfrak{m}$ gives that $B/\mathfrak{m}$ is actually a finite-dimensional vector space over $k$. But now $A/\phi^{-1}(\mathfrak{m})$ is isomorphic to a subspace of $B/\mathfrak{m}$ via the injection $\widetilde{\phi}$, so it is also a finite-dimensional vector space over $k$. ■