Extended and contracted ideals

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Let $\phi : A \to B$ be a ring homomorphism. For any ideal $\mathfrak{b}$ of B, its preimage $\phi^{-1}(\mathfrak{b})$ is an ideal of A, which we'll call its **contraction** and denote by $\mathfrak{b}^c$. On the other hand, given an ideal $\mathfrak{a}$ of A, its image by ϕ is not, in general, an ideal of B (it's only an ideal of the subring $\phi(A)$). We'll call the ideal of B generated by $\phi(\mathfrak{a})$ the **extension** of $\mathfrak{a}$, written $\mathfrak{a}^e$.

An ideal $\mathfrak{a}$ is said to be *contracted* if there exists some ideal $\mathfrak{b}$ such that $\mathfrak{a} = \mathfrak{b}^c$. Similarily, an ideal $\mathfrak{b}$ is said to be *extended* if there exists some ideal $\mathfrak{a}$ such that $\mathfrak{b} = \mathfrak{a}^e$.

These constructions form a Galois connection, with extension being the lower adjoint and contraction the upper adjoint. Recall that any Galois connection enjoys some formal properties:

- Its kernel operator is "contraction followed by extension", so we have $\mathfrak{b}^{ce} \leq \mathfrak{b}$ and $\mathfrak{b}^{cec} = \mathfrak{b}^c$.
- Its closure operator is "extension followed by contraction", therefore $\mathfrak{a} \leq \mathfrak{a}^{ec}$ and $\mathfrak{a}^e = \mathfrak{a}^{ece}$.
- As upper adjoints preserve meets, $(\mathfrak{b}_1 \cap \mathfrak{b}_2)^c = \mathfrak{b}_1^c \cap \mathfrak{b}_2^c$.
- Similarily, lower adjoints preserve joins: $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = \mathfrak{a}_1^e + \mathfrak{a}_2^e$.

In addition to those formal properties, this connection enjoys further algebraic relations. See Atiyah-Macdonald's book *Introduction to Commutative Algebra*, p.10, exercise 1.18, for many of them. For instance:

Supplementary relation 1: $(\mathfrak{a}_1\mathfrak{a}_2)^e = \mathfrak{a}_1^e\mathfrak{a}_2^e$**.**

First, let $x_1 \in \mathfrak{a}_1$ and $x_2 \in \mathfrak{a}_2$. Because products of the form x_1x_2 generate $\mathfrak{a}_1\mathfrak{a}_2$, and because $\phi(x_1)\phi(x_2) \in \mathfrak{a}_1^e\mathfrak{a}_2^e$, we find that $\phi(\mathfrak{a}_1\mathfrak{a}_2) \subseteq \mathfrak{a}_1^e\mathfrak{a}_2^e$, so we obtain the first inclusion, $(\mathfrak{a}_1\mathfrak{a}_2)^e \leq \mathfrak{a}_1^e\mathfrak{a}_2^e$.

Second, let $y_1 \in \mathfrak{a}_1^e$ and $y_2 \in \mathfrak{a}_2^e$. Then for $k = 1, 2$, there exists an integer $n \geq 0$, and for each $1 \leq i \leq n$ elements $x_{k,i} \in \mathfrak{a}_k$ and elements $b_k^i \in B$ such that, using Einstein's summation convention, $y_k = b_k^i\phi(x_{k,i})$. Hence the product y_1y_2 can be expressed as $b_1^ib_2^j\phi(x_{1,i}x_{2,j})$, which is clearly an element of $(\mathfrak{a}_1\mathfrak{a}_2)^e$. Because products of the form y_1y_2 generate $\mathfrak{a}_1^e\mathfrak{a}_2^e$, we obtain the other inclusion and we win.

Supplementary relation 2: $(\sqrt{\mathfrak{b}})^c = \sqrt{\mathfrak{b}^c}$.

It's really easy to prove this one by a simple chain of "if and only if"s, using the fact an element y is in the radical of $\mathfrak{b}$ if and only if there exists some integer $n > 0$ such that $y^n \in \mathfrak{b}$.

Applications of extended and contracted ideals to localization

We can apply these notions to better understand the ideal structure of the localization of a ring, in terms of the ideals in the original ring. From now on, we take ϕ to be the canonical ring homomorphism from A to its localization $S^{-1}A$ at some multiplicative submonoid S.

In this context, the contraction of an ideal is basically just singleling out all of the numerators, while extension is putting back all of the possible denumerators. Here's a more precise statement about extension: recall that S^{-1} is a functor that may also be applied to modules, not just rings. Any ideal $\mathfrak{a}$ of A is an A-module, so it makes sense to write $S^{-1}\mathfrak{a}$, which is just the set of fractions having as numerator an element of $\mathfrak{a}$ and as denumerator an element of S. By bringing terms to a common denumerator, we have:

$$\mathfrak{a}^e = S^{-1}\mathfrak{a}.$$

Proposition. For the ring homomorphism $A \to S^{-1}A$, the extension-contraction connection is a Galois insertion, that is, for each ideal $\mathfrak{b}$ in the localization, we have $\mathfrak{b}^{ce} = \mathfrak{b}$.

Proof. We always have $\mathfrak{b}^{ce} \leq \mathfrak{b}$. To prove the reverse inclusion, let a/s be an element of $\mathfrak{b}$. Now $a/1$ is in $\mathfrak{b}$ as well, and that element lies in the image of ϕ; in fact, $a/1 = \phi(a)$. Therefore, $a \in \phi^{-1}(\mathfrak{b}) = \mathfrak{b}^c$. This means that we have $a/1 \in \phi(\mathfrak{b}^c)$. By definition, $\mathfrak{b}^{ce}$ is the ideal generated by $\phi(\mathfrak{b}^c)$ hence $a/1 \in \mathfrak{b}^{ce}$. Multiplying by $1/s$ yields $a/s \in \mathfrak{b}^{ce}$. ■

Thus we see that in the context of localization, contraction is an injective operation, while extension is a surjective operation. In particular, every ideal in $S^{-1}A$ is an extended ideal.

Proposition. If $\mathfrak{p}$ is a prime ideal of A which doesn't meet S, then we have $\mathfrak{p}^{ec} = \mathfrak{p}$.

Proof. We always have $\mathfrak{p} \leq \mathfrak{p}^{ec}$. To prove the reverse inclusion, let x be an element of $\mathfrak{p}^{ec}$. Then $\phi(x) = x/1$ is an element of $\mathfrak{p}^e$. Since $\mathfrak{p}^e = S^{-1}\mathfrak{p}$, we can write $x/1$ as a fraction y/s where $y \in \mathfrak{p}$ and $s \in S$. Hence there exists $s' \in S$ such that $s'sx = s'y \in \mathfrak{p}$, so $x \in \mathfrak{p}$ by primality and the fact no element of S lies in $\mathfrak{p}$. ■

Corollary. Extension-contraction gives a bijective, inclusion-preserving correspondence between the set of prime ideals of A which don't meet S, and the set of prime ideals of $S^{-1}A$.

Proof. In view of the previous two propositions, we only have to prove that: (i) for any prime ideal $\mathfrak{q}$ of $S^{-1}A$, its contraction $\mathfrak{q}^c$ is also a prime ideal which doesn't meet S, and (ii) for any prime ideal $\mathfrak{p}$ of A which doesn't meet S, its extension $\mathfrak{p}^e$ is also a prime ideal. Note also that the inclusion-preserving part of the corollary is a formal consequence of Galois connections.

For (i), it's a general fact about ring homomorphism that the preimage $\mathfrak{q}^c$ is also a prime ideal. Now, if $\mathfrak{q}^c$ contained some element of S, then $\mathfrak{q}^{ce}$ would contain an invertible element, so would be the whole ring; by the first proposition we know that's not the case, so $\mathfrak{q}^c$ doesn't meet S.

For (ii), by primality of $\mathfrak{p}$ the ring $A/\mathfrak{p}$ is an integral domain. Let $\overline{S}$ denote the image of S in $A/\mathfrak{p}$ and consider the usual isomorphism of rings

$$\overline{S}^{-1}(A/\mathfrak{p}) \cong S^{-1}A/S^{-1}\mathfrak{p}.$$

Because $\mathfrak{p}$ doesn't meet S, the image $\overline{S}$ doesn't contain zero. This is enough to conclude the left hand ring is not the zero ring. Also, the left hand ring is contained in a field (the field of fractions of $A/\mathfrak{p}$), so its only zero divisor is 0. Therefore, the right hand side is a non-zero ring which is an integral domain, so $S^{-1}\mathfrak{p} = \mathfrak{p}^e$ is a prime ideal of $S^{-1}A$. ■

Further applications

- For some prime ideal $\mathfrak{p}$ of A, the prime ideals in the local ring $A_\mathfrak{p}$ corresponds via extension-contraction to the prime ideals in A which are contained in $\mathfrak{p}$.

- For any $f \in A$ which is not nilpotent, there exists a prime ideal of A which doesn't contain f. To see it, notice that the localized ring A_f is not the zero ring, so admits a maximal ideal. By the corollary, the contraction of this maximal ideal corresponds to a prime ideal in A which doesn't contain f.
  In geometric terms, we proved that if f is a function on $\operatorname{Spec} A$ that is not everywhere vanishing, then there exists a point at which f is not zero. Pretty obvious when you put it that way!

- Building on the idea of the previous point, we can show that the set of all nilpotents in a ring (its *nilradical*) is equal to the intersection of all prime ideals of the ring. Any nilpotent is obviously contained in every prime ideal since some power of it is zero. On the other hand, given an element which is not nilpotent, there exists some prime ideal that doesn’t contain it, hence in particular the element is not contained in the intersection of all prime ideals.