Reflexions on morphisms of affine schemes

written by rapha on Functor Network
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My goal is to write here some thoughts and insights about morphisms of affine schemes. Morphisms of affine schemes and morphism of (commutative, with 1) rings are dual, here I'm trying to gain intuition about this duality.

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Consider for instance the "intuitive" map $x \mapsto y = x^2$ from $\mathbb{C}$ to $\mathbb{C}$. To cast this in the language of schemes, we wish to define a "corresponding" morphism from $\operatorname{Spec} \mathbb{C}[x]$ to $\operatorname{Spec} \mathbb{C}[y]$. This morphism should "be" the same as the "intuitive" map. To do this, we specify a map of rings $\mathbb{C}[y] \to \mathbb{C}[x]$ (notice the direction), defined by $y \mapsto x^2$.

- Any polynomial in y becomes a polynomial in x^2 through this map. This effectively means x plays the role of the "square root" of y, so a polynomial such as $y - 4$ splits as $(x-2)(x+2)$.
- What I find confusing: I perceive the classical map $x \mapsto y = x^2$ from $\mathbb{C}$ to $\mathbb{C}$ as some kind of machine that, given a concrete value $x \in \mathbb{C}$, produces the concrete result $y = x^2$. Importantly, points can be multiplied, since they are just numbers.
- On the other hand, the induced set function $\operatorname{Spec} \mathbb{C}[x] \to \operatorname{Spec} \mathbb{C}[y]$ operates not on points but on ideals, and the image of $\mathfrak{p}$ is not the square of $\mathfrak{p}$, but $\phi^{-1}(\mathfrak{p})$, i.e. the set of polynomials such that, when I rename all instances of y to x^2, I get a polynomial that lies in $\mathfrak{p}$.
- I'm struggling to reconcile the two points of view. Their equivalence should be in the fact the induced map of affine schemes sends the maximal ideal (i.e. closed point) $(x-a)$ to the maximal ideal $(y-a^2)$.
- This is not too hard to show in this particular case. Indeed, it suffices to show that $y - a^2$ lies in $\phi^{-1}((x-a))$ because (i) the preimage of the maximal ideal $(x-a)$ is maximal, and (ii) the ideal $(y-a^2)$ is itself maximal. We have

  $$\phi(y - a^2) = x^2 - a^2 = (x-a)(x+a),$$

  so that we indeed have that $(x-a)$ is sent to $(y-a^2)$.
- But even with such a proof, I'm not grokking what's going on. For instance, how can we compute directly the image of $(x-a)$, without guesstimating beforehand that it should be $(y-a^2)$?
- I think maybe the answer to that last question is: fibers. If we are able to compute the fiber over each maximal ideal $(y-b)$, and if we can describe

them well, then to understand to which point is $(x-a)$ sent, we check in which fiber it is. This works because closed points are sent to closed points always. *May 29,2025: no, closed points are not always sent to closed points. I detail one such situation in another post.*

- In our specific case, we know $(x-\sqrt{b})$ and $(x+\sqrt{b})$ both map to $(y-b)$. But in general, what are the closed points of a fiber? Can we find this info from the fact the polynomial $y-b$ becomes x^2-b through the ring morphism, and so it can then be split further as $y-b=(x-\sqrt{b})(x+\sqrt{b})$, using the fact x plays the role of the square root of y? *Answer: yes, see next day*
- To extend the previous question, suppose the ring morphism is general (i.e. it maps y to an arbitrary polynomial f). Are the closed points in the fiber all of the form $(x-a)$ where a is a root of f? Tangential question: is it even possible that there are non-closed points in the fiber of a closed point, at all? In any case, are the points in the fiber, the irreducible factors of f?
- Some progress into insight: as in the previous point, let the map of rings ϕ send y to some arbitrary polynomial f. Then $\phi(y-f(a))$ evaluates to zero at the point a (that's trivial), or in other words $\phi(y-f(a)) \in (x-a)$. Thus $y-f(a)$ lies in $\phi^{-1}((x-a))$, so by maximality of the ideals involved we have that $\phi^{-1}((x-a))$ is equal to $(y-f(a))$. Therefore the point $(x-a)$ goes to $(y-f(a))$ via the induced map between spectra.

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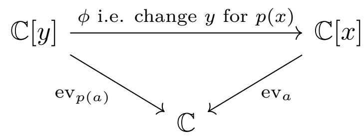
Continuing on the same example, I had an insight just before falling asleep yesterday. Consider that, by the Nullstellesatz, closed points are identified with maximal ideals, and those are of the form $(y-b)$ because $\mathbb{C}$ is algebraically closed. That means the point $b \in \mathbb{C}$ is identified with the set of all polynomials *which vanish when evaluated at* b.

Key insight, which seems really childish after the fact: now, evaluating a polynomial $f(y)$ at a^2 is clearly the same as first changing all occurences of y to x^2 and then evaluating at a. In other words,

$$f \in (y-a^2) \iff \phi(f) \in (x-a) \iff f \in \phi^{-1}((x-a)).$$

I'm pretty sure this convinces me that the induced map of schemes sends $a \in \mathbb{C}$ to $a^2 \in \mathbb{C}$ (i.e. it sends $(x-a)$ to $(y-a^2)$). In fact, it convinces me of the more general fact: let ϕ be the more general map of rings, sending y to a general polynomial $p(x)$. Then the induced map of schemes sends the point $a \in \mathbb{C}$ to $p(a) \in \mathbb{C}$.

- The evaluation map is what relates classical points and schematic points. More precisely, each closed schematic point is the kernel of the appropriate evaluation map.



- So then, $\ker(\mathrm{ev}_a \circ \phi) = \phi^{-1}((x-a))$ is equal to $\ker \mathrm{ev}_{p(a)} = (y - p(a))$.
- Again and in a slightly different way: obviously, $a \in \mathbb{C}$ is a root of some polynomial $q(p(x))$ if and only if $p(a)$ is a root of $q(y)$. That is, the ideal $(x-a)$ corresponds to the ideal $(y-p(a))$ via the renaming of y to $p(x)$.

This gives more insight, but I feel like I will ponder on this more when this has settled a bit in my thoughts.

What about my questions on fibers? Did sleep help? A little bit:

- For any ring morphism $\phi : A \to B$ and corresponding induced map of affine schemes $\pi : \operatorname{Spec} B \to \operatorname{Spec} A$, unraveling the definitions yields

$$\mathfrak{q} \in \pi^{-1}(\mathfrak{p}) \iff (f \in \mathfrak{p} \iff \phi(f) \in \mathfrak{q})$$

- In our initial example, this means a point $\mathfrak{q}$ of $\operatorname{Spec} \mathbb{C}[x]$ is in the fiber above $(y-b)$ if and only if $f(y) \in (y-b) \iff f(x^2) \in \mathfrak{q}$. In particular, taking $f = y - b$ gives us that $x^2 - b$ lies in $\mathfrak{q}$. Because $x^2 - b$ splits as $(x-\sqrt{b})(x+\sqrt{b})$, primality of $\mathfrak{q}$ implies one of those factors lie in it. Hence $\mathfrak{q}$ is either the ideal $(x-\sqrt{b})$ or $(x+\sqrt{b})$.
- More generally, over $\mathbb{C}$ every polynomial splits as a product of linear factors. So by the same logic the fiber above some point $(y-b)$ is going to be (as a set) the collection of maximal ideals generated by the linear factors of $p(x) - b$.