

Classic results on homogeneous polynomials

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Let A be any (commutative, with identity) ring. A polynomial f in the ring $A[x_1, \dots, x_n]$ is said to be **homogeneous of degree** $d \in \mathbb{N}$ if there exists elements $a_m \in A$ (with $m \in \mathbb{N}^n$, $|m| = d$, and not all a_m are zero) such that

$$f = \sum_{\substack{m \in \mathbb{N}^n \\ |m|=d}} a_m x^m$$

Homogeneous polynomials are important in algebraic geometry because they are used to build stuff in projective space. That's possible precisely because homogeneous polynomials are those which behave well under a scaling transformation of their variables:

Proposition. A polynomial $f \in A[x_1, \dots, x_n]$ is homogeneous of degree d if and only if the relation $f(tx_1, \dots, tx_n) = t^d f(x_1, \dots, x_n)$ holds between these elements of $A[x_1, \dots, x_n, t]$. ■

Notice that there's an induction principle for homogeneous polynomials of a given degree d : to prove a proposition is true about them, it suffices to show that it is true for monomials of degree d , and that the proposition is stable under sums of homogeneous polynomials of degree d . This technique may be used to easily prove the following:

Euler' Theorem. Let $F \in A[x_1, \dots, x_n]$ be an homogeneous polynomial of degree $k \geq 0$. Then

$$kF = \sum_{i=1}^n x_i \frac{\partial F}{\partial x_i}.$$

Proof. We proceed by Noetherian induction as indicated above. For the base case, suppose F is a monomial $F = cx^m$, with m a multi-index $m \in \mathbb{N}^n$ having $|m| = k$, and $c \in A$ a coefficient. Differentiating F with respect to some fixed variable x_i yields zero if $m_i = 0$, otherwise it yields $cm_i x^{m'}$ where m' is the multi-index obtained from m by decreasing its i th component by one. In both cases, multiplying the resulting expression by x_i gives the term $cm_i x^m$. Taking the sum of these terms for i from 1 to n and factoring out the common factor cx^m yields the desired equation.

For the induction step, suppose G and H are two homogeneous polynomials of degree k such that the equation holds for them; we want to show it also holds for their sum $G + H$. This is obvious, since the partial differential is a linear operator. ■

The following result generalizes Euler's Theorem (just take $s = 1$) and can be proven using the same technique:

Proposition. Let $F \in A[x_1, \dots, x_r]$ be a homogeneous polynomial of degree $n \geq 0$. For any natural number $s \geq 1$, we have

$$n(n-1) \cdots (n-s+1)F = \sum x_{i_1} x_{i_2} \cdots x_{i_s} \frac{\partial^s F}{\partial x_{i_1} \cdots \partial x_{i_s}}$$

where the sum is taken over all tuples $(i_1, \dots, i_s) \in \{1, \dots, r\}^s$. ■