Classic results on homogeneous polynomials

written by rapha on Functor Network
original link: https://functor.network/user/2593/entry/1076

Let A be any (commutative, with identity) ring. A polynomial f in the ring $A[x_1, \dots, x_n]$ is said to be **homogeneous of degree** $d \in \mathbb{N}$ if there exists elements $a_m \in A$ (with $m \in \mathbb{N}^n$, $|m| = d$, and not all a_m are zero) such that

$$f = \sum_{\substack{m \in \mathbb{N}^n \\ |m| = d}} a_m x^m$$

Homogeneous polynomials are important in algebraic geometry because they are used to build stuff in projective space. That's possible precisely because homogeneous polynomials are those which behave well under a scaling transformation of their variables:

Proposition. A polynomial $f \in A[x_1, \dots, x_n]$ is homogeneous of degree d if and only if the relation $f(tx_1, \dots, tx_n) = t^d f(x_1, \dots, x_n)$ holds between these elements of $A[x_1, \dots, x_n, t]$. ■

Notice that there's an induction principle for homogeneous polynomials of a given degree d: to prove a proposition is true about them, it suffices to show that it is true for monomials of degree d, and that the proposition is stable under sums of homogeneous polynomials of degree d. This technique may be used to easily prove the following:

Euler' Theorem. Let $F \in A[x_1, \dots, x_n]$ be an homogeneous polynomial of degree $k \geq 0$. Then

$$kF = \sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i}.$$

Proof. We proceed by Noetherian induction as indicated above. For the base case, suppose F is a monomial $F = cx^m$, with m a multi-index $m \in \mathbb{N}^n$ having $|m| = k$, and $c \in A$ a coefficient. Differentiating F with respect to some fixed variable x_i yields zero if $m_i = 0$, otherwise it yields $cm_i x^{m'}$ where m' is the multi-index obtained from m by decreasing its ith component by one. In both cases, multiplying the resulting expression by x_i gives the term $cm_i x^m$. Taking the sum of these terms for i from 1 to n and factoring out the common factor cx^m yields the desired equation.

For the induction step, suppose G and H are two homogeneous polynomials of degree k such that the equation holds for them; we want to show it also holds for their sum $G + H$. This is obvious, since the partial differential is a linear operator. ■

The following result generalizes Euler's Theorem (just take $s = 1$) and can be proven using the same technique:

Proposition. Let $F \in A[x_1, \ldots, x_r]$ be a homogeneous polynomial of degree $n \geq 0$. For any natural number $s \geq 1$, we have

$$n(n-1)\cdots(n-s+1)F = \sum x_{i_1} x_{i_2} \cdots x_{i_s} \frac{\partial^s F}{\partial x_{i_1} \cdots \partial x_{i_s}}$$

where the sum is taken over all tuples $(i_1, \ldots, i_s) \in \{1, \ldots, r\}^s$. ■